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LONG CIRCUITS AND LARGE EULER SUBGRAPHS*

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Abstract. We study the parameterized complexity of the following Euler subgraph problems: (a) LARGE EULER SUBGRAPH: For a given graph G and integer parameter k, does G contain an induced Eulerian subgraph with at least k vertices? (b) LONG CIRCUIT: For a given graph G and integer parameter k, does G contain an Eulerian subgraph with at least k edges? Our main algorithmic result is that LARGE EULER SUBGRAPH is fixed parameter tractable (FPT) on undirected graphs. The complexity of the problem changes drastically on directed graphs, and we obtain the following complexity dichotomy: LARGE EULER SUBGRAPH is NP-hard for every fixed $k > 3$ and is solvable in polynomial time for $k \leq 3$. For LONG CIRCUIT, we prove that the problem is FPT on directed and undirected graphs.

Key words. Euler subgraph, long circuit, parameterized complexity

AMS subject classifications. 05C85, 68R10, 68Q17, 68Q25, 68W40

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1. Introduction.

One of the oldest theorems in graph theory is attributed to Euler, and it says that a (undirected) graph admits an *Euler circuit*, i.e., a closed walk visiting every edge exactly once, if and only if the graph is connected and all its vertices are of even degrees. Respectively, a directed graph has a *directed Euler circuit* if and only if the graph is (weakly) connected and for each vertex its in-degree is equal to its out-degree. While checking if a given directed or undirected graph is Eulerian is easily done in polynomial time, the problem of finding k edges (arcs) in a graph to form an Eulerian subgraph is NP-hard. We refer to the book of Fleischner [12] for a thorough study of Eulerian graphs and related topics.

In [5], Cai and Yang initiated the study of parameterized complexity of subgraph problems motivated by Eulerian graphs. In particularly, they considered the following parameterized subgraph and induced subgraph problems:

k-CIRCUIT	**Parameter:** k
Input: A (directed) graph G and nonnegative integer k	
Question: Does G contain a circuit with k edges (arcs)?	

and

EULER k-SUBGRAPH	**Parameter:** k
Input: A (directed) graph G and nonnegative integer k	
Question: Does G contain an induced Euler subgraph with k vertices?	

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Clearly, a graph has a circuit with k edges (arcs) if and only if G has an Euler subgraph with k edges (arcs), i.e., k-Circuit asks about existence of an Euler subgraph with k edges (arcs).

The nonparameterized versions of both k-Circuit and Euler k-Subgraph are known to be NP-complete [5]. Cai and Yang in [5] proved that k-Circuit on undirected graphs is fixed parameter tractable (FPT). On the other hand, the authors have shown in [14] that Euler k-Subgraph is W[1]-hard. The variant of the problem $(m-k)$-Circuit, where one asks to remove at most k edges to obtain an Eulerian subgraph, was shown to be FPT by Cygan et al. [7] on directed and undirected graphs. The problem of removing at most k vertices to obtain an induced Eulerian subgraph, namely, Euler $(n-k)$-Subgraph, was shown to be W[1]-hard by Cai and Yang for undirected graphs [5] and by Cygan et al. for directed graphs [7]. Dorn et al. in [8] provided FPT algorithms for the weighted version of Eulerian extension.

In this work we extend the set of results on the parameterized complexity of Eulerian subgraph problems by considering the problems of finding an (induced) Eulerian subgraph with *at least* k (vertices) edges. We consider the following problems:

Large Euler Subgraph **Parameter:** k
Input: A (directed) graph G and nonnegative integer k
Question: Does G contain an induced Euler subgraph with at least k vertices?

and

Long Circuit **Parameter:** k
Input: A (directed) graph G and nonnegative integer k
Question: Does G contain a circuit with at least k edges (arcs)?

Using the observation of Cygan et al. in [7], it is not difficult to see that the nonparameterized versions of Long Circuit and Large Euler Subgraph are NP-complete for directed and undirected graphs. Let us note that by plugging these observations into the framework of Bodlaender et al. [4], it is easy to conclude that both problems have no polynomial kernels unless NP $\subseteq$ coNP /poly.

However, the question about the parameterized complexity of these problems appears to be much more interesting.

Our results. We show that Large Euler Subgraph behaves differently for directed and undirected cases. For undirected graphs, we prove that the problem is FPT. We find it a bit surprising, because the very closely related Euler k-Subgraph problem is known to be W[1]-hard [14]. The proof is based on a structural result interesting on its own. Roughly speaking, we show that large treewidth certifies containment of a large induced Euler subgraph. For directed graphs, Large Euler Subgraph is NP-complete for each $k \geq 4$, and this bound is tight—the problem is polynomial-time solvable for each $k \leq 3$. We also prove that Euler k-Subgraph is W[1]-hard for directed graphs. Long Circuit is proved to be FPT for directed and undirected graphs. Our algorithm is based on the results by Gabow and Nie [17] about the parameterized complexity of finding long cycles. The known and new results about Euler subgraph problems are summarized in Table 1.

This paper is organized as follows. Section 2 contains basic definitions and preliminaries. In section 3.1 we show that Large Euler Subgraph is FPT on undirected graphs. In section 3.2 we prove that on directed graphs, Euler k-Subgraph is W[1]-hard while Large Euler Subgraph is NP-complete for each $k \geq 4$. In section 4 we show that Long Curcuit is FPT on directed and undirected graphs.

Table 1
Parameterized complexity of Euler subgraph problems.

	Undirected	Directed
k-Circuit	FPT [5]	FPT, Prop. 4.4
Euler k-Subgraph	W[1]-hard [14]	W[1]-hard, Thm. 3.7
$(m-k)$-Circuit	FPT [7]	FPT [7]
Euler $(n-k)$-Subgraph	W[1]-hard [5]	W[1]-hard [7]
Long Circuit	FPT, Thm. 4.7	FPT, Cor. 4.3
Large Euler Subgraph	FPT, Thm. 3.6	NP-complete $\forall\ k \geq 4$, Thm. 3.8; in P for $k \leq 3$

2. Basic definitions and preliminaries. Graphs. We consider finite directed and undirected graphs without loops or multiple edges. The vertex set of a (directed) graph G is denoted by $V(G)$, and the edge set of an undirected graph and the arc set of a directed graph G are denoted by $E(G)$. To distinguish edges and arcs, the edge with two end-vertices u, v is denoted by $\{u, v\}$, and we write (u, v) for the corresponding arc. For an arc $e = (u, v)$, v is the *head* of e and u is the tail. For a set of vertices $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G induced by S, and by $G - S$ we denote the graph obtained from G by the removal of all the vertices of S, i.e., the subgraph of G induced by $V(G)\backslash S$. Let G be an undirected graph. For a vertex v, we denote by $N_G(v)$ its *(open) neighborhood*, that is, the set of vertices which are adjacent to v. The *degree* of a vertex v is denoted by $d_G(v) = |N_G(v)|$, and $\Delta(G)$ is the maximum degree of G. Let now G be a directed graph. For a vertex $v \in V(G)$, we say that u is an *in-neighbor* of v if $(u, v) \in E(G)$. The set of all in-neighbors of v is denoted by $N^-_G(v)$. The *in-degree* $d^-_G(v) = |N^-_G(v)|$. Respectively, u is an *out-neighbor* of v if $(v, u) \in E(G)$, the set of all out-neighbors of v, is denoted by $N^+_G(v)$, and the *out-degree* $d^+_G(v) = |N^+_G(v)|$.

For a (directed) graph G, a (directed) *trail* of *length* k is defined as a sequence $v_0, e_1, v_1, e_2, \ldots, e_k, v_k$ of vertices and edges (arcs, resp.) of G such that $v_0, \ldots, v_k \in V(G)$, $e_1, \ldots, e_k \in E(G)$, the edges (arcs, resp.) $e_1, \ldots, e_k$ are pairwise distinct, and for $i \in \{1, \ldots, k\}$, $e_i = \{v_{i-1}, v_i\}$ ($e_i = (v_{i-1}, v_i)$, resp.). A trail is said to be *closed* if $v_0 = v_k$. A closed (directed) trail is called a (directed) *circuit*, and it is a (directed) *cycle* if all its vertices except $v_0 = v_k$ are distinct. Clearly, every cycle is a subgraph of G, and it is said that C is an *induced cycle* of G if $C = G[V(C)]$. A (directed) path is a trail such that all its vertices are distinct. For a (directed) walk (trail, path, resp.) $v_0, e_1, v_1, e_2, \ldots, e_k, v_k$, v_0 and v_k are its *end-vertices*, and $v_1, \ldots, v_{k-1}$ are its *internal* vertices. For a (directed) walk (trail, path, resp.) with end-vertices u and v, we say that it is an (u, v)-*walk* (*trail*, *path*, resp.). We omit the word "directed" if it does not create confusion. Also we write a trail as a sequence of its vertices $v_0, \ldots, v_k$.

A connected (directed) graph G is an *Euler* (or *Eulerian*) graph if it has a (directed) circuit that contains all edges (arcs, resp.) of G. By the celebrated result of Euler (see, e.g., [12]), a connected graph G is an Euler graph if and only if all its vertices have even degrees. Respectively, a connected directed graph G is an Euler directed graph if and only if for each vertex $v \in V(G)$, $d^-_G(v) = d^+_G(v)$.

Ramsey numbers. The *Ramsey number* $R(r, s)$ is the minimal integer n such that any graph on n vertices has either a clique of size r or an independent set of size s. By the famous paper of Erdös and Szekeres [10], $R(r, s) \leq \binom{r+s-2}{r-1}$.

Parameterized complexity. Parameterized complexity is a two-dimensional framework for studying the computational complexity of a problem. One dimension is the input size n and another one is a parameter k. It is said that a problem is FPT if it can be solved in time $f(k) \cdot n^{O(1)}$ for some function f, and it is said that a problem is in XP if it can be solved in time $O(n^{f(k)})$ for some function f. One of the basic assumptions of the parameterized complexity theory is the conjecture that the complexity class W[1] $\neq$ FPT, and it is unlikely that a W[1]-hard problem could be solved in FPT time. A problem is Para-NP-*hard*(*complete*) if it is NP-hard (complete) for some fixed value of the parameter k. Clearly, a Para-NP-hard problem is not in XP unless P=NP. We refer to the books of Downey and Fellows [9], Flum and Grohe [13], and Niedermeier [22] for detailed introductions on parameterized complexity.

Treewidth. A *tree decomposition* of an undirected graph G is a pair (X, T), where T is a tree and $X = \{X_i \mid i \in V(T)\}$ is a collection of subsets (called *bags*) of $V(G)$ such that

1. $\bigcup_{i\in V(T)} X_i = V(G)$,
2. for each edge $\{x, y\} \in E(G)$, $x, y \in X_i$ for some $i \in V(T)$, and
3. for each $x \in V(G)$ the set $\{i \mid x \in X_i\}$ induces a connected subtree of T.

The *width* of a tree decomposition $(\{X_i \mid i \in V(T)\}, T)$ is $\max_{i\in V(T)} \{|X_i| - 1\}$. The *treewidth* of a graph G (denoted as $\mathbf{tw}(G)$) is the minimum width over all tree decompositions of G.

We conclude this section with simple observations about the hardness of the considered problems. The results of Cygan et al. in [7] immediately imply that the nonparameterized version of Long Circuit is NP-complete. By a similar argument we obtain the same for Large Euler Subgraph.

Proposition 2.1. Long Circuit *and* Large Euler Subgraph *are* NP-*complete for directed and undirected graphs when* k *is a part of the input.*

Proof. Let G be an n-vertex undirected cubic graph. It is straightforward to see that G has a circuit with at least n edges if and only if G is Hamiltonian. As the Hamiltonian Cycle is known to be NP-complete for cubic planar graphs [18], it follows that Long Circuit is NP-complete for undirected graphs. Denote by G' the graph obtained by subdividing each edge of G. Now we observe that G' has an induced Euler subgraph with at least $2n$ vertices if and only if G is Hamiltonian. We have that Large Euler Subgraph is NP-complete for undirected graphs.

For directed graphs, we use similar arguments. Let G be a directed graph. Denote by G' the graph obtained from G by the replacement of each vertex $v \in V(G)$ by two vertices v^+, v^- joined by an arc (v^+, v^-), and we replace each arc $(u, v) \in E(G)$ by (u^-, v^+). Because G has a circuit with at least $2n$ edges if and only if G is Hamiltonian and because Hamiltonian Cycle is NP-complete for directed graphs [18], Long Circuit is NP-complete for directed graphs. Finally, let G'' be the directed graph obtained by subdividing each arc of G'. Since G'' has an induced Euler subgraph with at least $4n$ vertices if and only if G is Hamiltonian, we conclude that Large Euler Subgraph is NP-complete for directed graphs. ∎

Observe that if a (directed) graph G has components $G_1, \ldots, G_t$, then G has a circuit of size k (an induced Euler subgraph with at least k vertices, resp.) if and only if there is $i \in \{1, \ldots, t\}$ such that G_i has a circuit of size k (an induced Euler subgraph with at least k vertices, resp.). By this observation, Proposition 2.1, and the results by Bodlaender et al. [4], we have the following claim. We refer to [9, 13, 22] for the definition of a polynomial kernel.

PROPOSITION 2.2. LONG CIRCUIT *and* LARGE EULER SUBGRAPH *for directed and undirected graphs have no polynomial kernels unless* NP $\subseteq$ coNP /poly.

3. Large Euler subgraphs.

3.1. Large Euler subgraphs for undirected graphs. In this section we show that LARGE EULER SUBGRAPH is FPT for undirected graphs. Using Ramsey arguments, we prove that if a graph G has sufficiently large treewidth, then G has an induced Euler subgraph on at least k vertices. Then if the input graph has large treewidth, we have a YES-answer. Otherwise, we use the fact that LARGE EULER SUBGRAPH is FPT parameterized by the treewidth of a graph. All graphs considered here are undirected.

For a given positive integer k, we define the function $f(\ell)$ for integers $\ell \geq 2$ recursively as follows:

- $f(2) = R(k, k-1) + 1$,
- $f(\ell) = (k-1)(2(\ell-1)(f(\lfloor \frac{\ell}{2} \rfloor + 1) - 1) + 1) + 1$ for $\ell > 2$.

We need the following two lemmas.

LEMMA 3.1. *Let G be a graph, and suppose that s, t are distinct vertices joined by at least $f(\ell)$ internally vertex-disjoint paths of length at most ℓ in G for some $\ell \geq 2$. Then G has an induced Euler subgraph on at least k vertices.*

Proof. Consider the minimum value of ℓ such that G has $f(\ell)$ internally vertex disjoint (s,t)-paths of length at most ℓ. We have at least $r = f(\ell) - 1$ such paths $P_1, \ldots, P_r$ that are distinct from the trivial (s,t)-path with one edge. We assume that each path P_i has no chords that either join two internal vertices or an internal vertex and one of the end-vertices, i.e., each internal vertex is adjacent in $G[V(P_i)]$ only to its two neighbors in P_i. Otherwise, we can replace P_i by a shorter path with all vertices in $V(P_i)$ distinct from the path s, t. We consider two cases.

Case 1. $\ell = \mathbf{2}$. The paths $P_1, \ldots, P_r$ are of length two and therefore have exactly one internal vertex. Assume that $u_1, \ldots, u_r$ are internal vertices of these paths. Because $r = f(2) - 1 = R(k, k-1)$, the graph $G[\{u_1, \ldots, u_r\}]$ either has a clique K of size k or an independent set I of size at least $k-1$. Suppose that G has a clique K. If k is odd, then $G[K]$ is an induced Euler subgraph on k vertices. If k is even, then $G[K \cup \{s\}]$ is an induced Euler subgraph on $k+1$ vertices. Assume now that $I \subseteq \{u_1, \ldots, u_{r-1}\}$ is an independent set of size $k-1$. Let $v \in I$. If $\{s,t\} \in E(G)$ and k is even or $\{s,t\} \notin E(G)$ and k is odd, then $G[I \cup \{s,t\}]$ is an induced Euler subgraph on $k+1$ vertices. Else if $\{s,t\} \notin E(G)$ and k is even or $\{s,t\} \in E(G)$ and k is odd, then $G[I \cup \{s,t\} \setminus \{v\}]$ is an induced Euler subgraph on k vertices.

Case 2. $\ell \geq \mathbf{3}$. We say that paths P_i and P_j are *adjacent* if they have adjacent internal vertices. Let $p = f(\lfloor \ell/2 \rfloor + 1)$. Suppose that there is an internal vertex v of one of the paths $P_1, \ldots, P_r$ that is adjacent to at least $2p-1$ internal vertices of some other distinct $2p-1$ paths. Then there are $p = f(\lfloor \ell/2 \rfloor + 1)$ paths $P_{i_1}, \ldots, P_{i_p}$ that have respective internal vertices $v_1, \ldots, v_p$ such that

(i) v is adjacent to $v_1, \ldots, v_p$, and
(ii) either each v_j is at distance at most $\lfloor \ell/2 \rfloor$ from s in P_{i_j} for all $j \in \{1, \ldots, p\}$ or each v_j is at distance at most $\lfloor \ell/2 \rfloor$ from t in P_{i_j} for all $j \in \{1, \ldots, p\}$.

But then either the vertices s, v or v, t are joined by at least $f(\lfloor \ell/2 \rfloor + 1)$ internally vertex-disjoint paths of length at most $\lfloor \ell/2 \rfloor + 1 < \ell$. This contradicts our choice of ℓ. Hence, for each $i \in \{1, \ldots, r\}$, every internal vertex of P_i is adjacent to internal vertices in at most $2p-2$ other paths, and P_i is adjacent to at most $2(\ell-1)(p-1)$ other paths. As $r = (k-1)(2(\ell-1)(p-1)+1)$, there are $k-1$ distinct paths $P_{i_1}, \ldots, P_{i_{k-1}}$ that are pairwise nonadjacent, i.e., they have no adjacent internal vertices.

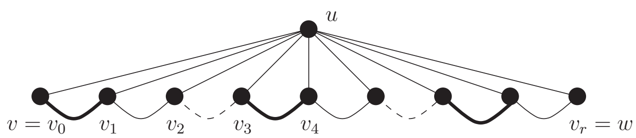

FIG. 1. *The path P and the graphs Q_1 (shown by the thick lines), Q_2 (shown by the thin lines), and Q_3 (shown by the dashed lines).*

Let $H = G[V(P_{i_1}) \cup \cdots \cup V(P_{i_{k-1}})]$ and $H' = G[V(P_{i_1}) \cup \cdots \cup V(P_{i_{k-2}})]$. Notice that by our choice of the paths, $H = P_{i_1} \cup \cdots \cup P_{i_{k-1}}$ and $H' = P_{i_1} \cup \cdots \cup P_{i_{k-2}}$ if $\{s,t\} \notin E(G)$, and $P_{i_1} \cup \cdots \cup P_{i_{k-1}}$ ($P_{i_1} \cup \cdots \cup P_{i_{k-2}}$, resp.) can be obtained from H (H', resp.) by the removal of $\{s,t\}$ if s,t are adjacent. If $\{s,t\} \in E(G)$ and k is even or $\{s,t\} \notin E(G)$ and k is odd, then H is an induced Euler subgraph on at least $k+1$ vertices. Else if $\{s,t\} \notin E(G)$ and k is even or $\{s,t\} \in E(G)$ and k is odd, then H' is an induced Euler subgraph on at least k vertices. ☐

Now we show that if a 2-connected graph G has a vertex of sufficiently large degree, then we can find an induced Euler subgraph on at least k vertices using Lemma 3.1. For $k \geq 4$, let

$$\Delta_k = 1 + \frac{(f(3k-8)-1)((f(3k-8)-2)^{3(k-3)}-1)}{f(3k-8)-3}.$$

LEMMA 3.2. *For $k \geq 4$, any 2-connected graph G with $\Delta(G) > \Delta_k$ has an induced Euler subgraph on at least k vertices.*

Proof. Let G be a 2-connected graph and let u be a vertex of G with $d_G(u) = \Delta(G)$. As G is 2-connected, $G' = G - u$ is connected. Let v be an arbitrary vertex of $N_G(u)$. Denote by T a tree of shortest paths from v to all other vertices of $N_G(u)$ in G', i.e., T is a tree in G' such that for any $w \in N_G(w)$, the unique (v,w)-path in T is a shortest (v,w)-path in G'.

Claim 1. If there is a (v,w)-path P of length at least $3(k-3)+1$ in T for some $w \in N_G(u)$, then G has an induced Euler subgraph on at least k vertices.

Proof. Let P be a (v,w)-path P of length at least $3(k-3)+1$ in T. Denote by $v_0,\ldots,v_r$ the vertices of P in $N_G(u)$. We assume that they are enumerated according to the order in which they occur in P tracing it from v to w. In particular $v_0 = v$ and $w = v_r$. We consider the $(v_0,v_1),\ldots,(v_{r-1},v_r)$-subpaths of P. We construct the graph Q_1 by taking unions of every third subpath starting from the (v_0,v_1)-subpath, Q_2 is constructed by taking every third subpath starting from the (v_1,v_2)-subpath, and Q_3 is obtained when we start from the (v_2,v_3)-subpath. Formally, Q_1 is the union of the $(v_0,v_1),(v_3,v_4),\ldots,(v_{3\lfloor r/3\rfloor},v_{3\lfloor r/3\rfloor+1})$-subpaths of P, Q_2 is the union of the $(v_1,v_2),(v_4,v_5),\ldots,(v_{3\lfloor r/3\rfloor+1},v_{3\lfloor r/3\rfloor+2})$-subpaths of P, and Q_3 is the union of the $(v_2,v_3),(v_5,v_6),\ldots,(v_{3\lfloor r/3\rfloor-1},v_{3\lfloor r/3\rfloor})$-subpaths of P, as shown in Figure 1. Notice that some subpaths can be empty depending on whether r modulo 3 is 0, 1, or 2. By the constructions, Q_1, Q_2, Q_3 are edge-disjoint. Because T is a shortest path tree, we have that Q_1, Q_2, Q_3 are induced subgraphs of G. Since $Q_1 \cup Q_2 \cup Q_3 = P$, there is Q_i for $i \in \{1,2,3\}$ with at least $k-2$ edges. Then Q_i has at least $k-1$ vertices. Let $H = G[V(Q_i) \cup \{u\}]$. By the definition of Q_i, H is a union of induced cycles with one common vertex u such that for different cycles C_1, C_2 in the union, $V(C_1) \cap V(C_2) = \{u\}$ and $\{x,y\} \notin E(G)$ whenever $x \in V(C_1)\backslash\{u\}$ and $y \in V(C_2)\backslash\{u\}$. Hence, H is an Euler graph with at least k vertices. ☐

From now we assume that all (v,w)-paths in T have length at most $3(k-3)$ for $w \in N_G(u)$.

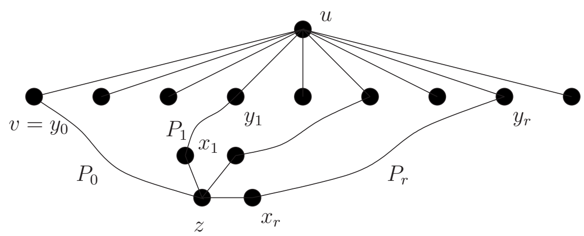

FIG. 2. *The paths $P_0, \dots, P_r$.*

Claim 2. If there is a vertex $z \in V(T)$ with $d_T(z) \geq f(3(k-3)+1)$, then G has an induced Euler subgraph on at least k vertices.

Proof. Recall that T is a tree of shortest paths from v to all other vertices of $N_G(u)$ in G'. We assume that T is rooted in v. Then the root defines the parent-child relation on T. Let $x_1, \dots, x_r$ be the children of z. If z has no parent, then $z = v$ and $r \geq f(3(k-3)+1)$. Otherwise, $r \geq f(3(k-3)+1) - 1$. Let $y_0 = v$. Because each leaf of T is a vertex of $N_G(u)$, for each $i \in \{1, \dots, r\}$, there is a closest descendant $y_i \in V(T) \cap N_G(u)$ of x_i in T. Denote by P_i the unique (z, y_i)-path in T for $i \in \{0, \dots, r\}$, as shown in Figure 2. As all (v, w)-paths in T have length at most $3(k-3)$ for $w \in N_G(u)$, the paths $P_0, \dots, P_r$ have length at most $3(k-3)$. Notice that these paths have no common vertices except z. Observe now that $y_0, \dots, y_r$ are adjacent to u in G. Therefore, we have at least $f(3(k-3)+1)$ internally vertex-disjoint (u, z)-paths in G. By Lemma 3.1, it implies that G has an induced Euler subgraph on at least k vertices. ∎

To complete the proof of the lemma, it remains to observe that if $\Delta(T) < f(3(k-3)+1)$ and all (v, w)-paths in T have length at most $3(k-3)$ for $w \in N_G(u)$, then

$$d_G(u) \leq |V(T)| \leq 1 + \frac{(f(3k-8)-1)((f(3k-8)-2)^{3(k-3)}-1)}{f(3k-8)-3} = \Delta_k. \qquad \square$$

Kosowski et al. [20] obtained the following bound for treewidth.

THEOREM 3.3 (see [20]). *Let $k \geq 3$ and let G be a graph without induced cycles of length at least k and $\Delta(G) \geq 1$. Then $\mathbf{tw}(G) \leq k(\Delta(G)-1)+2$.*

This theorem together with Lemma 3.2 immediately imply the next lemma.

LEMMA 3.4. *Let G be a graph and let $k \geq 4$. If $\mathbf{tw}(G) > k(\Delta_k - 1) + 2$, then G has an induced Euler subgraph on at least k vertices.*

Proof. Suppose that $\mathbf{tw}(G) > k(\Delta_k - 1) + 2$. It is well known that the treewidth of a graph G is equal to the maximum treewidth of its 2-connected components. Then G has a 2-connected component G' with $\mathbf{tw}(G') > k(\Delta_k - 1) + 2$. If $\Delta(G') > \Delta_k$, then G' has an induced Euler subgraph on at least k vertices by Lemma 3.2. Otherwise, by Theorem 3.3, G' has an induced cycle on at least k vertices, i.e., an induced Euler subgraph. ∎

Now we observe that LARGE EULER SUBGRAPH is FPT for graphs of bounded treewidth.

LEMMA 3.5. *For any positive integer t,* LARGE EULER SUBGRAPH *can be solved in linear time for graphs of treewidth at most t.*

Proof. Recall that the syntax of the *monadic second-order logic* (MSOL) of graphs includes logical connectives $\vee$, $\wedge$, $\neg$, variables for vertices, edges, sets of vertices and edges, and quantifiers $\forall$, $\exists$ that can be applied to these variables. Besides the standard relations $=$, $\in$, $\subseteq$, the syntax includes the relation $\mathrm{adj}(u,v)$ for two vertex variables, which expresses whether two vertices u and v are adjacent, and for a vertex variable v and an edge variable e, we have the relation $\mathrm{inc}(v,e)$ which expresses that v is incident with e. The *counting monadic first-order logic* (CMSOL) is an extension of MSOL with the additional predicate $\mathrm{card}_{p,q}(X)$ which expresses whether the cardinality of a set X is p modulo q.

By the celebrated Courcelle's theorem, any problem that can be expressed in MSOL can be solved in linear time for graphs of bounded treewidth. Moreover, this result holds also for optimization problems that can be expressed in CMSOL (see, e.g., the monograph of Courcelle and Engelfriet [6]).

Observe that to solve Large Euler Subgraph for a graph G, it is sufficient to find a subset of vertices U of maximum size such that U induces an Euler graph. Clearly, U induces an Euler graph if and only if (i) $G[U]$ is connected and (ii) each vertex of $G[G]$ has even degree. The both properties can be expressed in CMSOL. The standard way to express connectivity is to notice that $G[U]$ is connected if and only if for any $X \subset U$, $X \neq \emptyset$, and $X \neq U$, there is an edge $\{x,y\} \in E(G)$ such that $x \in X$ and $y \in U \setminus X$. Then we have to express the property that for any $u \in U$, $d_{G[U]}(u) = |N_{G[U]}(u)|$ is even. To do it, it is sufficient to observe that $X = N_{G[U]}(u)$ if and only if $X \subseteq U$ such that (i) for any $v \in X$, $\{u,v\} \in E(G)$, and (ii) for any $v \in U$ such that $\{u,v\} \in E(G)$, $v \in X$. Since we can express in CMSOL whether $|N_{G[U]}(u)|$ is even, the claim follows. ∎

Now we can prove the main result of this section.

Theorem 3.6. *For any positive integer k,* Large Euler Subgraph *can be solved in linear time for undirected graphs.*

Proof. Clearly, we can assume that $k \geq 3$, as any Euler graph has at least three vertices. If $k = 3$, then we can find any shortest cycle in the input graph G. It is straightforward to see that if G has no cycles, then we have no Euler subgraph, and any induced cycle is an induced Euler subgraph on at least three vertices. Hence, it can be assumed that $k \geq 4$. We check in linear time whether $\mathbf{tw}(G) \leq k(\Delta_k - 1) + 2$ using the Bodlaender's algorithm [3]. If it is so, we solve our problem using Lemma 3.5. Otherwise, by Lemma 3.4, we conclude that G has an induced Euler subgraph on at least k vertices and return a YES-answer. ∎

Notice that the proof of Theorem 3.6 is not constructive. Next, we sketch the algorithm that produces an induced Euler subgraph on at least $k \geq 4$ vertices if it exists.

First, for each $\ell \geq 2$, we can test the existence of two vertices s,t such that the input graph G has at least $f(\ell)$ internally vertex-disjoint (s,t)-paths of length at most ℓ in FPT time with the parameter ℓ using the color coding technique [19]. If we find such a structure for $\ell \leq 3k - 8$, we find an induced Euler subgraph with at least k vertices that is either a clique or a union of (s,t)-paths, as explained in the proof of Lemma 3.1.

Otherwise, we find all 2-connected components. If there is a 2-connected component G' with a vertex u with $d_{G'}(u) > \Delta_k$, then we find an induced Euler subgraph with at least k vertices that is a union of cycles with the common vertex u using the arguments form the proof of Lemma 3.2.

If all 2-connected components have bounded maximum degrees, we use the algorithm of Kosowski et al. [20] that in polynomial time either finds an induced cycle on at least k vertices or constructs a tree decomposition of width at most $k(\Delta_k - 1) + 2$. In the first case we have an induced Euler subgraph on at least k vertices. In the second case the treewidth is bounded, and LARGE EULER SUBGRAPH is solved by a dynamic programming algorithm instead of applying Lemma 3.5.

3.2. Large Euler subgraphs for directed graphs. In this section we show that EULER k-SUBGRAPH and LARGE EULER SUBGRAPH are hard for directed graphs.

First, we consider EULER k-SUBGRAPH. It is straightforward to see that this problem is in XP, since we can check for every subset of k vertices whether it induces an Euler subgraph. We prove that this problem cannot be solved in FPT time unless FPT = W[1].

THEOREM 3.7. *The* EULER k-SUBGRAPH *is* W[1]*-hard for directed graphs.*

Proof. We reduce the MULTICOLORED k-CLIQUE problem that is known to be W[1]-hard [11]:

MULTICOLORED k-CLIQUE **Parameter:** k
Input: A k-partite graph $G = (V_1 \cup \cdots \cup V_k, E)$, where $V_1, \dots, V_k$ are sets of the k-partition
Question: Does G has a clique with k vertices?

Let $G = (V_1 \cup \cdots \cup V_k, E)$ be an instance of MULTICOLORED k-CLIQUE. We construct the directed graph H as follows:

- Construct the copies of $V_1, \dots, V_k$.
- For $1 \le i < j \le k$ and for each $u \in V_i$ and $v \in V_j$, if $\{u, v\} \notin E(G)$, then construct an arc (u, v) for the copies of u and v in H.
- For each $i \in \{1, \dots, k\}$, construct two vertices x_i, y_i, then join x_i by arcs with all vertices of V_i and join every vertex of V_i with y_i by an arc.
- Construct arcs $(y_1, x_2), (y_2, x_3), \dots, (y_k, x_1)$.

We set $k' = 3k$.

We claim that G has a clique with k vertices if and only if H has an induced Euler subgraph with at least k' vertices.

Let $K = \{v_1, \dots, v_k\}$ be a clique in G where $v_i \in V_i$ for $i \in \{1, \dots, k\}$. Observe that $v_1, \dots, v_k$ are pairwise nonadjacent in H. Hence, the set of vertices $\{x_1, v_1, y_1, \dots, x_k, v_k, y_k\}$ induces a cycle in H. Hence, we have an induced Euler subgraph with at least k' vertices.

Suppose now that H has an induced Euler subgraph C with at least k' vertices. Observe that every directed cycle in G contains the arc (y_k, x_1), because if we delete this arc, we obtain a directed acyclic graph. Since any Euler directed graph is a union of arc-disjoint directed cycles (see, e.g., [12]), C is an induced directed cycle. Moreover, for each $i \in \{1, \dots, k\}$, C contains at most one vertex of V_i. Indeed, assume that two vertices u, v of C are in the same set V_i. Then the (u, v)-paths and the (v, u)-path in C should contain (y_k, x_1), but this is impossible. Because C has $3k$ vertices, we conclude that C contains exactly one vertex v_i from each V_i for $i \in \{1, \dots, k\}$, and $x_i, y_i \in V(C)$ for $i \in \{1, \dots, k\}$. Then $C = H[\{x_1, v_1, y_1, \dots, x_k, v_k, y_k\}]$. Since C is an induced cycle, $v_1, \dots, v_k$ are pairwise nonadjacent in H. Then $\{v_1, \dots, v_k\}$ is a clique in G. ∎

For LARGE EULER SUBGRAPH for directed graphs, we prove that this problem is Para-NP-complete.

THEOREM 3.8. *For any* $k \ge 4$, LARGE EULER SUBGRAPH *is* NP*-complete for oriented graphs, i.e., for directed graphs without cycles of length two.*

Proof. We reduce the 3-SATISFIABILITY problem. It is known [2] that this problem is NP-complete even if each variable is used in exactly 4 clauses: two clauses contain the literal x_i and two clauses contain the literal $\overline{x}_i$.

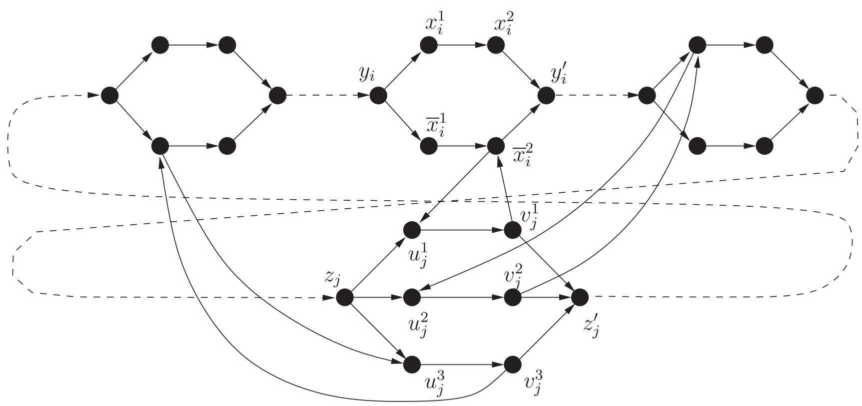

FIG. 3. *Construction of G.*

Let $k \geq 4$ be an integer. Suppose that Boolean variables $x_1, \dots, x_n$ and clauses $C_1, \dots, C_m$ compose an instance of 3-SATISFIABILITY such that each variable is used in exactly 4 clauses: 2 times in positive and 2 times in negations. Let $\phi = C_1 \wedge \cdots \wedge C_m$. Without loss of generality we assume that $k \leq 4(n+m)$. We construct the oriented graph G as follows (see Figure 3):

- For each $i \in \{1, \dots, n\}$, construct vertices $y_i, y_i', x_i^1, x_i^2, \overline{x}_i^1, \overline{x}_i^2$ and arcs (y_i, x_i^1), $(x_i^1, x_i^2), (x_i^2, y_i')$, $(y_i, \overline{x}_i^1)$, $(\overline{x}_i^1, \overline{x}_i^2), (\overline{x}_i^2, y_i')$.
- For each $i \in \{2, \dots, n\}$, construct (y_{i-1}', y_i).
- For each $j \in \{1, \dots, m\}$, construct vertices $z_j, z_j', u_j^1, u_j^2, u_j^3, v_j^1, v_j^2, v_j^3$ and arcs (z_j, u_j^1), (u_j^1, v_j^1), (v_j^1, z_j'), (z_j, u_j^2), (u_j^2, v_j^2), (v_j^2, z_j'), (z_j, u_j^3), $(u_j^3, v_j^3), (v_j^3, z_j')$.
- For each $j \in \{2, \dots, m\}$, construct (z_{j-1}', z_j).
- Construct $(y_n', z_1), (z_m', y_1)$.
- For each $j \in \{1, \dots, m\}$, let $C_j = w_1 \vee w_2 \vee w_3$. For $h \in \{1, 2, 3\}$,
  - if $w_h = x_i$ for some $i \in \{1, \dots, n\}$ and w_h is the first occurrence of the literal x_i in ϕ, then construct $(x_i^1, u_j^h), (v_j^h, x_i^1)$;
  - if $w_h = x_i$ for some $i \in \{1, \dots, n\}$ and w_h is the second occurrence of the literal x_i in ϕ, then construct $(x_i^2, u_j^h), (v_j^h, x_i^2)$;
  - if $w_h = \overline{x}_i$ for some $i \in \{1, \dots, n\}$ and w_h is the first occurrence of the literal $\overline{x}_i$ in ϕ, then construct $(\overline{x}_i^1, u_j^h), (v_j^h, \overline{x}_i^1)$;
  - if $w_h = \overline{x}_i$ for some $i \in \{1, \dots, n\}$ and w_h is the second occurrence of the literal $\overline{x}_i$ in ϕ, then construct $(\overline{x}_i^2, u_j^h), (v_j^h, \overline{x}_i^2)$.

We claim that ϕ can be satisfied if and only if G has an induced Euler subgraph with at least k vertices.

For $i \in \{1, \dots, n\}$, let $P_i = \{y_i, x_i^1, x_i^2, y_i'\}$ and $\overline{P}_i = \{y_i, \overline{x}_i^1, \overline{x}_i^2, y_i'\}$. For $j \in \{1, \dots, n\}$ and $h \in \{1, 2, 3\}$, $Q_j^h = \{z_j, u_j^h, v_j^h, z_j'\}$. Let also $Z = \{z_1, \dots, z_m\}$ and $Z' = \{z_1', \dots, z_m'\}$.

Suppose that the variables $x_1, \dots, x_n$ have values such that ϕ is satisfied. We construct the set of vertices U as follows:

- for $i \in \{1, \dots, n\}$, if $x_i = true$, then the vertices of the set $\overline{P}_i$ are included in U, and P_i is included otherwise;
- for $j \in \{1, \dots, m\}$, if $C_j = w_1 \vee w_2 \vee w_3$, then we choose a literal $w_h = true$ in C_j and include Q_j^h in U.

Observe that U induces a cycle in G and $|U| = 4(n+m) \geq k$. Then $G[U]$ is an induced Euler subgraph on at least k vertices.

Assume now that a set $U \subseteq V(G)$ induces an Euler graph $H = G[U]$ and $|U| \geq k$. Notice that because $|U| \geq k \geq 4$, for every vertex $w \in U$, U contains at least one in-neighbor of w and at least one out-neighbor of w. Also the number of in-neighbors and the number of out-neighbors of w in U is the same.

Since $k \geq 4$, $U \cap (Z \cup Z') \neq \emptyset$ because $G - (Z \cup Z')$ has no cycles except vertex-disjoint triangles. Hence, there is $j \in \{1, \ldots, m\}$ such that $z_j \in U$ or $z'_j \in U$. Suppose that $z_j \in U$. If $j = 1$, then $y'_n \in U$ because it is the unique in-neighbor of z_1, and if $j > 1$, then $z'_{j-1} \in U$ as $z'_{j-1} \in U$ is the unique in-neighbor. Then exactly one of the out-neighbors of z_j, i.e., exactly one of the vertices u_j^1, u_j^2, u_j^3, is in U. Let $u_j^h \in U$. Then its unique out-neighbors $v_j^h \in U$. Further, exactly one of the out-neighbors of v_j^h is in U, and either $z'_j \in U$ or some vertex $x_i^p \in U$ or $\overline{x}_i^p \in U$. But if $x_i^p \in U$ or $\overline{x}_i^p \in U$, $d_H^-(u_j^h) = 2 > 1 = d_H^+(u_j^h)$. Hence, $z'_j \in U$. If $j = m$, then $y_1 \in U$, and if $j < m$, then $z_{j+1} \in U$. We use similar arguments for the case $z'_j \in U$. If $j = m$, then $y_1 \in U$, and if $j < m$, then $z_{j+1} \in U$. Also, exactly one of the vertices v_j^1, v_j^2, v_j^3 is in U. Let $v_j^h \in U$. Then $u_j^h \in U$. Further, either $z_j \in U$ or some vertex $x_i^p \in U$ or $\overline{x}_i^p \in U$ because U contains an in-neighbor of $u_j^h \in U$. But if $x_i^p \in U$ or $\overline{x}_i^p \in U$, $d_H^-(v_j^h) = 1 < 2 = d_H^+(v_j^h)$. Hence, $z_j \in U$. If $j > 1$, then $z'_{j-1} \in U$. We have that for each $j \in \{1, \ldots, m\}$, exactly one $Q_h \subseteq U$ for $h \in \{1, 2, 3\}$, and for $h' \in \{1, 2, 3\} \backslash \{h\}$, $u_j^{h'}, v_j^{h'} \notin U$. Also we have that $y_1 \in U$.

Now we consequently consider $i = 1, \ldots, n$. We already know that $y_1 \in U$. Assume inductively that $y_i \in U$. Then exactly one of the vertices $x_i^1, \overline{x}_i^1$ is in U. Assume that x_i^1 is in U as another case is symmetric. Then either x_i^2 or some vertex u_j^h should be in U. But if $u_j^h \in U$, then because $z_j \in U$, $d_H^-(u_j^h) = 2 > d_H^+(u_j^h)$ and this is impossible for an Euler graph. Then $x_j^2 \in U$. By the same arguments we show that $y'_i \in U$. If $i < n$, then $y_{i+1} \in U$, and we can proceed with our inductive arguments. We conclude that for each $i \in \{1, \ldots, n\}$, either $P_i \subseteq U$ or $\overline{P}_i \subseteq U$, and if $P_i \subseteq U$ ($\overline{P}_i \subseteq U$, resp.), then $\overline{x}_i^1, \overline{x}_i^2 \notin U$ ($x_i^1, x_i^2 \notin U$, resp.). Moreover, if $P_i \subseteq U$ ($\overline{P}_i \subseteq U$, resp.) and $Q_j^h \subseteq U$, then H has no arcs between the vertices of these two sets.

We define the truth assignment for the variables $x_1, \ldots, x_n$ as follows: for each $i \in \{1, \ldots, n\}$, if $P_i \subseteq U$, then $x_i = false$, and $x_i = true$ otherwise. We claim that ϕ is satisfied by this assignment. To show it, consider a clause $C_j = w_1 \vee w_2 \vee w_3$. We know that there is $h \in \{1, 2, 3\}$ such that $Q_j^h \subseteq U$. Assume that $w_h = x_i$ (the case $w_h = \overline{x}_i$ is symmetric). Then Q_i^h is joined by arcs in G with P_i. It follows that P_i was not included in U, i.e., $\overline{P}_i \subseteq U$ and $x_i = true$. Hence, $w_h = true$. ∎

We proved that Large Euler Subgraph is NP-complete for directed graphs for $k \geq 4$. In the conclusion of this section we observe that the bound $k \geq 4$ is tight unless P = NP.

Proposition 3.9. Large Euler Subgraph *can be solved in polynomial time for* $k \leq 3$.

Proof. For $k = 1$, the problem is trivial. If $k = 2$, then any shortest cycle C in a directed graph G is an induced Euler subgraph of G with at least two vertices, and G has no induced Euler subgraph if G is a directed acyclic graph. Hence, it remains to consider $k = 3$.

Suppose that H is an induced Euler subgraph of a directed graph G, and H has at least three vertices. Denote by H' the graph obtained from H by the deletion of all pairs of opposite arcs, i.e., for each pair of vertices x, y such that $(x, y), (y, x) \in E(H)$, we delete $(x, y), (y, x)$. Clearly, for any $v \in V(H')$, $d_{H'}^-(v) = d_{H'}^+(v)$. If H' is empty, then because $|V(H)| \geq 3$, there are three distinct vertices $x, y, z \in V(H)$ such that $(x, y), (y, x), (y, z), (z, y) \in E(H)$ and either $(x, z), (z, x) \in E(H)$ or $(x, z), (z, x) \notin E(H)$. Then $G[\{x, y, z\}]$ is an Euler subgraph of G. If H' is nonempty, then H' has a shortest cycle C. Because H' has no cycles of length two, $G[V(C)]$ is an induced Euler subgraph with at least three vertices.

We conclude that LARGE EULER SUBGRAPH can be solved for $k = 3$ as follows. If G has three distinct vertices x, y, z such that $(x, y), (y, x), (y, z), (z, y) \in E(G)$ and either $(x, z), (z, x) \in E(G)$ or $(x, z), (z, x) \notin E(G)$, then $G[\{x, y, z\}]$ is an Euler subgraph of G. Otherwise, let G' be the graph obtained from G by the deletion of all pairs of opposite arcs. We find a shortest cycle C in G', and we have that $G[V(C)]$ is an induced Euler subgraph with at least three vertices. Finally, if G' is a directed acyclic graph, then we return a NO-answer. ∎

4. Long circuits. In this section we show that the LONG CIRCUIT problem is FPT for directed and undirected graphs. Our algorithms are based on known results about the LONG CYCLE problem:

LONG CYCLE	**Parameter:** k
Input: A (directed) graph G and a positive integer k	
Question: Does G contain a cycle with at least k edges (arcs)?	

Fomin, Lokshtanov, and Saurabh [15, 16] gave a first single-exponential FPT algorithm for the problem on directed graphs. In particular, they proved the following theorem.

THEOREM 4.1 (Theorem 5.1 of [16]). *The* LONG CYCLE *problem can be solved in time* $8^{k+o(k)} \cdot mn^2 \log n$ *for directed graphs with* n *vertices and* m *arcs.*

Let G be a directed graph. Recall that the *line digraph* $L(G)$ of G is a directed graph that has the vertex set $E(G)$, and for any two distinct arcs $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ of G, (e_1, e_2) is an arc of $L(G)$ if and only if $v_2 = u_1$. We use the following folklore observation and provide its proof for completeness.

LEMMA 4.2. *For a directed graph* G *and a positive integer* k*,* G *has a directed circuit with at least* k *arcs if and only if* $L(G)$ *has a directed cycle on at least* k *vertices.*

Proof. Suppose that G has a directed circuit $v_0, e_1, v_1, e_2, \ldots, e_r, v_r$. Then, by the definition of $L(G)$, the vertices $e_1, \ldots, e_r, e_1$ of $L(G)$ compose a directed cycle. Suppose now that $L(G)$ has a directed cycle $e_0, \ldots, e_r$. For each $i \in \{0, \ldots, r-1\}$, e_i and e_{i+1} have a common incident vertex v_i such that v_i is the head of e_i and is the tail of e_{i+1}. Then $v_0, e_1, \ldots, e_r, v_0$ is a directed circuit in G. ∎

Theorem 4.1 and Lemma 4.2 immediately provide the following corollary.

COROLLARY 4.3. *The* LONG CIRCUIT *problem can be solved in time* $8^{k+o(k)} \cdot m^4 \log n$ *for directed graphs with* n *vertices and* m *arcs.*

Observe also that Lemma 4.2 together with the results of Fomin, Lokshtanov, and Saurabh [16] implies the following proposition. Notice that its undirected analogue was proved by Cai and Yang in [5].

PROPOSITION 4.4. *The* k-CIRCUIT *problem can be solved in time* $O(2.851^k \cdot m \log^2 n)$ *for directed graphs with* n *vertices and* m *arcs.*

For undirected graphs, it is slightly more convenient to use the structural result by Gabow and Nie [17]. Let us recall that a fundamental cycle in an undirected graph is formed from a spanning tree and a nontree edge.

THEOREM 4.5 (see [17]). *In a connected undirected graph G having a cycle with at least k edges, either there is a fundamental cycle with at least k edges for every depth-first search tree or some cycle in G with at least k edges has at most $2k-4$ edges.*

We need the following observation.

LEMMA 4.6. *Let G be a graph without cycles of length at least k. If G has a circuit with at least k edges, then G has a circuit with at least k and at most $2k-2$ edges.*

Proof. Let C be a circuit in G. It is well known (see, e.g., [12]) that C is a union of edge-disjoint cycles $C_1, \ldots, C_r$. Moreover, it can be assumed that for any $i \in \{1, \ldots, r\}$, the graph $C_1 \cup \cdots \cup C_i$ is connected, i.e., it is a circuit. Suppose now that C is a circuit with at least k edges that has minimum length. Then the circuit $C' = C_1 \cup \cdots \cup C_{r-1}$ has at most $k-1$ edges. Since G has no cycles of length at least k, C_r has at most $k-1$ edges. Thus, C has at most $2k-2$ edges. ☐

Now we are ready to prove the FPT-result for LONG CIRCUIT on undirected graphs.

THEOREM 4.7. *The* LONG CIRCUIT *problem can be solved in $O(k(2e)^{2k} \cdot nm)$ expected time and in $(2e)^{2k}k^{O(\log k)} \cdot nm \log n$ worst-case time for graphs with n vertices and m edges.*

Proof. We assume that the input graph G is connected, as otherwise we can solve the problem for each component. We choose a vertex v arbitrarily and perform the depth-first search from v. In this way we find the fundamental cycles for the dfs-tree rooted in v and check whether there is a fundamental cycle of length at least k. If we have such a cycle, then it is a circuit with at least k edges, and we have a YES-answer. Otherwise, by Theorem 4.5, either G has no cycles of length at least k or G has a cycle with at least k and at most $2k-4$ edges. Hence, if G has a cycle of length at least k, then G has a circuit with at least k and at most $2k-4$ edges. If G has no cycles with at least k edges, then by Lemma 4.6, if G has a circuit with at least k edges, it contains a circuit with at least k and at most $2k-2$ edges. We conclude that if the constructed fundamental cycles have lengths at most $k-1$, then G either has a circuit with at least k and at most $2k-2$ edges or has no circuit with at least k edges.

We check whether G has a circuit with at least k and at most $2k-2$ edges using the color coding technique proposed by Alon, Yuster, and Zwick [1]. Our algorithm is a variant of the algorithm of Cai and Yang [5] for k-CIRCUIT. Hence, we only sketch it here. For simplicity, we solve the decision problem, but the algorithm can be easily modified to obtain a circuit of prescribed size if it exists.

Let G be a graph with n vertices and m arcs.

First, we describe the randomized algorithm. We color the edges of G by $k' = 2k-2$ colors $1, \ldots, k'$ uniformly at random independently from each other. Now we are looking for a *colorful* circuit in G that has at least k edges, i.e., for a circuit such that all edges are colored by distinct colors.

To do it, we apply the dynamic programming across subsets. We choose an initial vertex u and try to construct a circuit that includes u. For a set of colors $X \subseteq \{1, \ldots, k'\}$, denote by $U(X)$ the set of vertices $v \in V(G)$ such that there is a (u, v)-trail with $|X|$ edges colored by distinct colors from X. It is straightforward to see that $U(\emptyset) = \{u\}$. For $X \neq \emptyset$, $v \in U(X)$ if and only if v has a neighbor $w \in N_G(v)$ such that $\{w, v\}$ is colored by a color $c \in X$ and $w \in U(X \backslash \{c\})$. We consequently construct the sets $U(X)$ for X with $1, 2, \ldots, k'$ elements. We stop and return a YES-answer if $u \in U(X)$ for some X of size at least k. Notice that the sets $U(X)$ can be constructed in time $O(k'2^{k'} \cdot m)$. Since we try all possibilities to select u, the running time is $O(k'2^{k'} \cdot mn)$.

Now we observe that for any positive number $p < 1$, there is a constant c_p such that after running our randomized algorithm $c_p e^{k'}$ times, we either get a YES-answer or can claim that with probability p G has no directed circuit with at least k and at most k' arcs.

This algorithm can be derandomized by the technique proposed by Alon, Yuster, and Zwick [1] using the k'-perfect hash functions constructed by Naor, Schulman, and Srinivasan [21]. To do it, we replace random colorings by the family of at most $e^{k'} k^{\log k} \log n$ hash functions.

Since the depth-first search runs in linear time, we have that LONG CIRCUIT can be solved in $O(k(2e)^{2k} \cdot nm)$ expected time and in $O((2e)^{2k} k^{O(\log k)} \cdot nm \log n)$ worst-case time. ∎

5. Conclusions. In this paper, we provide a complete classification of the parameterized complexity of different Euler subgraph problems; see Table 1. It is natural to ask whether it is possible to obtain better running times for the problems that are shown to be FPT. For LONG CIRCUIT on directed graphs, we can observe that by applying the representative set techniques proposed by Fomin, Lokshtanov, and Saurabh [16] directly to the problem, i.e., without transforming an instance of LONG CIRCUIT to the instance of LONG CYCLE for the line graph of the input graph using Lemma 4.2, we can obtain a better dependence on the input size. For the undirected case, we can solve the problem in time $O^*(8^{k+o(k)})$. For LARGE EULER SUBGRAPH, the dependence of the running time of our algorithm on the parameter is huge. Is it possible to solve the problem in single-exponential $O^*(2^{O(k)})$ time?

In a related question, an induced Euler subgraph of maximum size can be found in time $O^*(2^n)$ by a trivial brute-force algorithm trying all possible vertex subsets and checking if the induced subgraph is Eulerian. Can brute force be avoided here? In other words, is there time $O^*((2-\varepsilon)^n)$ algorithm for this problem?

REFERENCES

[1] N. ALON, R. YUSTER, AND U. ZWICK, *Color-coding*, J. ACM, 42 (1995), pp. 844–856.
[2] P. BERMAN, M. KARPINSKI, AND A. D. SCOTT, *Approximation hardness of short symmetric instances of MAX-3SAT*, Electronic Colloquium on Computational Complexity (2003).
[3] H. L. BODLAENDER, *A linear-time algorithm for finding tree-decompositions of small treewidth*, SIAM J. Comput., 25 (1996), pp. 1305–1317.
[4] H. L. BODLAENDER, R. G. DOWNEY, M. R. FELLOWS, AND D. HERMELIN, *On problems without polynomial kernels*, J. Comput. System Sci., 75 (2009), pp. 423–434.
[5] L. CAI AND B. YANG, *Parameterized complexity of even/odd subgraph problems*, J. Discrete Algorithms, 9 (2011), pp. 231–240.
[6] B. COURCELLE AND J. ENGELFRIET, *Graph Structure and Monadic Second-Order Logic—A Language-Theoretic Approach*, Encyclopedia Math. Appl. 138, Cambridge University Press, Cambridge, UK, 2012.
[7] M. CYGAN, D. MARX, M. PILIPCZUK, M. PILIPCZUK, AND I. SCHLOTTER, *Parameterized complexity of Eulerian deletion problems*, Algorithmica, 68 (2014), pp. 41–61.
[8] F. DORN, H. MOSER, R. NIEDERMEIER, AND M. WELLER, *Efficient algorithms for Eulerian extension and rural postman*, SIAM J. Discrete Math., 27 (2013), pp. 75–94.
[9] R. G. DOWNEY AND M. R. FELLOWS, *Fundamentals of Parameterized Complexity*, Texts in Comput. Sci., Springer, New York, 2013.
[10] P. ERDÖS AND G. SZEKERES, *A combinatorial problem in geometry*, Compos. Math., 2 (1935), pp. 463–470.

[11] M. R. Fellows, D. Hermelin, F. A. Rosamond, and S. Vialette, *On the parameterized complexity of multiple-interval graph problems*, Theoret. Comput. Sci., 410 (2009), pp. 53–61.

[12] H. Fleischner, *Eulerian Graphs and Related Topics, Part* 1*, Volume* 1, Ann. Discrete Math. 45, Elsevier, Amsterdam, 1990.

[13] J. Flum and M. Grohe, *Parameterized Complexity Theory*, Texts in Theoret. Comput. Sci., Springer-Verlag, Berlin, 2006.

[14] F. V. Fomin and P. A. Golovach, *Parameterized complexity of connected even/odd subgraph problems*, J. Comput. System Sci., 80 (2014), pp. 157–179.

[15] F. V. Fomin, D. Lokshtanov, and S. Saurabh, *Efficient Computation of Representative Sets with Applications in Parameterized and Exact Algorithms*, CoRR abs/1304.4626, 2013.

[16] F. V. Fomin, D. Lokshtanov, and S. Saurabh, *Efficient computation of representative sets with applications in parameterized and exact algorithms*, in Proceedings of SODA 14, C. Chekuri, ed., SIAM, Philadelphia, 2014, pp. 142–151.

[17] H. N. Gabow and S. Nie, *Finding a long directed cycle*, ACM Trans. Algorithms, 4 (2008).

[18] M. R. Garey, D. S. Johnson, and R. E. Tarjan, *The planar Hamiltonian circuit problem is NP-complete*, SIAM J. Comput., 5 (1976), pp. 704–714.

[19] P. A. Golovach and D. M. Thilikos, *Paths of bounded length and their cuts: Parameterized complexity and algorithms*, Discrete Optim., 8 (2011), pp. 72–86.

[20] A. Kosowski, B. Li, N. Nisse, and K. Suchan, *k-chordal graphs: From cops and robber to compact routing via treewidth*, in ICALP 2012, Lecture Notes in Comput. Sci., 7392, Springer, New York, 2012, pp. 610–622.

[21] M. Naor, L. J. Schulman, and A. Srinivasan, *Splitters and near-optimal derandomization*, in Proceedings of FOCS, IEEE Computer Society, 1995, pp. 182–191.

[22] R. Niedermeier, *Invitation to Fixed-Parameter Algorithms*, Oxford Lecture Ser. Math. Appl. 31, Oxford University Press, Oxford, UK, 2006.