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Parameterized complexity of connected even/odd subgraph problems ☆

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ABSTRACT

In 2011, Cai an Yang initiated the systematic parameterized complexity study of the following set of problems around Eulerian graphs: for a given graph G and integer k, the task is to decide if G contains a (connected) subgraph with k vertices (edges) with all vertices of even (odd) degrees. They succeed to establish the parameterized complexity of all cases except two, when we ask about:

- a connected k-edge subgraph with all vertices of odd degrees, the problem known as k-Edge Connected Odd Subgraph; and
- a connected k-vertex induced subgraph with all vertices of even degrees, the problem known as k-Vertex Eulerian Subgraph.

We show that k-Edge Connected Odd Subgraph is FPT and k-Vertex Eulerian Subgraph is W[1]-hard. Our FPT algorithm is based on a novel combinatorial result on the treewidth of minimal connected odd graphs with even amount of edges.

1. Introduction

One of the oldest theorems in Graph Theory is attributed to Euler, and it says that a graph admits an Euler walk, i.e. a walk visiting every edge exactly once, if and only if the graph is connected and all its vertices are of even degrees. While checking if a given graph is Eulerian, i.e. is connected and has no vertices of odd degrees, is easily done in polynomial time, the problem of finding k edges in a graph to form an Eulerian subgraph is NP-hard. We refer to the book of Fleischner [7] for a thorough study of Eulerian graphs and related topics.

An *even* graph (respectively, *odd* graph) is a graph with each vertex of an even (odd) degree. Thus an Eulerian graph is a connected even graph. Let Π be one of the following four graph classes: Eulerian graphs, even graphs, odd graphs, and connected odd graphs. In [4], Cai and Yang initiated the study of parameterized complexity of subgraph problems motivated by Eulerian graphs. For each Π, they defined the following parameterized subgraph and induced subgraph problems:

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Table 1
Parameterized complexity of k-Edge Π Subgraph and k-Vertex Π Subgraph.

	Eulerian	Even	Odd	Connected Odd
k-Edge	FPT [4]	FPT [4]	FPT [4]	FPT Theorem 2
k-Vertex	W[1]-hard Theorem 3	FPT [4]	FPT [4]	FPT [4]

> k-Edge Π Subgraph (resp. k-Vertex Π Subgraph)
>
> *Instance*: A graph G and non-negative integer k.
> *Parameter*: k.
> *Question*: Does G contain a subgraph with k edges from Π (resp. an induced subgraph on k vertices from Π)?

Cai and Yang established the parameterized complexity of all variants of the problem except k-Edge Connected Odd Subgraph and k-Vertex Eulerian Subgraph, see Table 1. It was conjectured that k-Edge Connected Odd Subgraph is FPT and k-Vertex Eulerian Subgraph is W[1]-hard. We resolve these open problems and confirm both conjectures.

The remaining part of the paper is organized as follows. In Section 2, we provide definitions and give preliminary results. In Section 3, we show that k-Edge Connected Odd Subgraph is FPT. Our algorithmic result is based on an upper bound for the treewidth of a minimal connected odd graphs with an even number of edges. We show that the treewidth of such graphs is always at most 3. The proof of this combinatorial result, which we find interesting in its own, is non-trivial and is given in Section 4. The bound on the treewidth is tight—complete graph on four vertices K_4 is a minimal connected odd graph with an even number of edges and its treewidth is 3. In Section 5, we prove that k-Vertex Eulerian Subgraph is W[1]-hard and observe that the problem remains W[1]-hard if we ask about (not necessarily induced) Eulerian subgraph on k vertices. We conclude the paper in Section 6 with some open problems.

2. Definitions and preliminary results

Graphs. We consider finite undirected graphs without loops or multiple edges. The vertex set of a graph G is denoted by $V(G)$ and its edge set by $E(G)$. A set $S \subseteq V(G)$ of pairwise adjacent vertices is called a *clique*. For a vertex v, we denote by $N_G(v)$ its *(open) neighborhood*, that is, the set of vertices which are adjacent to v. The distance between two vertices $u, v \in V(G)$ (i.e., the length of the shortest (u,v)-path in the graph) is denoted by $\mathrm{dist}_G(u,v)$. For a vertex $v \in V(G)$ and a set of vertices $S \subseteq V(G)$, the distance between v and S is $\mathrm{dist}_G(v,S) = \min\{\mathrm{dist}_G(v,u) | u \in S\}$. For a vertex v and a positive integer k, $N_G^{(k)}[v] = \{u \in V(G) \mid \mathrm{dist}_G(u,v) \leqslant k\}$. The *degree* of a vertex v is denoted by $d_G(v)$, and $\Delta(G)$ is the maximum degree of G. For a set of vertices $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G induced by S, and by $G - S$ we denote the graph obtained form G by the removal of all the vertices of S, i.e. the subgraph of G induced by $V(G) \setminus S$.

Parameterized complexity. Parameterized complexity is a two dimensional framework for studying the computational complexity of a problem. One dimension is the input size n and another one is a parameter k. It is said that a problem is *fixed parameter tractable* (or FPT), if it can be solved in time $f(k) \cdot n^{O(1)}$ for some function f. One of basic assumptions of the Parameterized Complexity theory is the conjecture that the complexity class W[1] $\neq$ FPT, and it is unlikely that a W[1]-hard problem could be solved in FPT-time. We refer to the books of Downey and Fellows [6], Flum and Grohe [8], and Niedermeier [9] for detailed introductions to parameterized complexity.

Treewidth. A *tree decomposition* of a graph G is a pair (X, T) where T is a tree and $X = \{X_i \mid i \in V(T)\}$ is a collection of subsets (called *bags*) of $V(G)$ such that:

1. $\bigcup_{i \in V(T)} X_i = V(G)$,
2. for each edge $\{x, y\} \in E(G)$, $x, y \in X_i$ for some $i \in V(T)$, and
3. for each $x \in V(G)$ the set $\{i \mid x \in X_i\}$ induces a connected subtree of T.

The *width* of a tree decomposition $(\{X_i \mid i \in V(T)\}, T)$ is $\max_{i \in V(T)}\{|X_i| - 1\}$. The *treewidth* of a graph G (denoted as $\mathbf{tw}(G)$) is the minimum width over all tree decompositions of G.

Minimal odd graphs with even number of edges. Let r be a vertex of G. We assume that G is *rooted* in r. Let G be a connected odd graph with an even number of edges. We say that G is a *minimal* if G has no proper connected odd subgraph with an even number of edges containing r.

The importance of minimal odd subgraphs with even numbers of edges is crucial for our algorithm because of the following combinatorial result.

Theorem 1. *Let G be a minimal connected odd graph with an even number of edges with a root r. Then* $\mathbf{tw}(G) \leqslant 3$.

For non-rooted graphs, we also have the following corollary.

Corollary 1. *For any minimal connected odd graph G with an even number of edges,* $\mathbf{tw}(G) \leqslant 3$.

Let us remark that the bound in Theorem 1 is tight—complete graph K_4 with a root vertex r is a minimal odd graph with even number of edges and of treewidth 3. The proof of Theorem 1 is given in Section 4. This proof is non-trivial and technical, and we find the combinatorial result of Theorem 1 to be interesting in its own. From the algorithmic perspective, Theorem 1 is a cornerstone of our algorithm; combined with color-coding technique of Alon, Yuster and Zwick from [1] it implies that k-Edge Connected Odd Subgraph is FPT. We give this algorithm in the next section.

3. Algorithm for k-Edge Connected Odd Subgraph

To give an algorithm for k-Edge Connected Odd Subgraph, in addition to Theorem 1, we also need the following result of Alon, Yuster and Zwick from [1] obtained by the color-coding technique.

Proposition 1. *(See [1].) Let H be a graph on k vertices with treewidth t and let G be an n-vertex graph. A subgraph of G isomorphic to H, if one exists, can be found in $O(2^{O(k)} \cdot n^{t+1})$ expected time and in $O(2^{O(k)} \cdot n^{t+1} \cdot \log n)$ worst-case time.*

We are ready to prove the main algorithmic result of this paper.

Theorem 2. *k-Edge Connected Odd Subgraph can be solved in time $O(2^{O(k \log k)} \cdot n^4 \cdot \log n)$ for n-vertex graphs.*

Proof. Let (G, k) be an instance of the problem. We apply the following algorithm.

Step 1. If k is odd and the maximum vertex degree $\Delta(G) \geqslant k$, then return Yes. Else if k is odd but $\Delta(G) < k$, then go to Step 3.

Step 2. If k is even and $\Delta(G) \geqslant k$, then we enumerate all odd connected graphs H with k edges of treewidth at most 3. For each odd graph H of treewidth at most 3 and with k edges, we use Proposition 1 to check whether G has a subgraph isomorphic to H. The algorithm returns Yes if such a graph H exists. Otherwise, we construct a new graph G by removing from the old graph G all vertices of degree at least k.

Step 3. Let us remark that at this step all vertices of the input graph G are of degree at most $k-1$. For each vertex v, check whether there is a connected odd subgraph H of G with k edges containing v. The algorithm returns Yes if a connected odd subgraph H with k edges exists for some vertex v, and it returns No otherwise.

In what follows we discuss the correctness of the algorithm and evaluate its running time.

If k is odd and $\Delta(G) \geqslant k$, then the star $K_{1,k}$ is a subgraph of G. Hence, G has a connected odd subgraph with k edges.

Let k be even and let $r \in V(G)$ be a vertex with $d_G(r) \geqslant k$. If G has a connected odd subgraph with k edges containing r, then G has a minimal connected odd subgraph H with even number of edges rooted in r. Let $\ell = |E(H)|$. Graph H contains at most ℓ vertices in $N_G(r)$. It follows that there are $k-\ell$ vertices $v_1, \dots, v_{k-\ell} \in N_G(r) \setminus V(H)$. Denote by H' the subgraph of G with the vertex set $V(H) \cup \{v_1, \dots, v_{k-\ell}\}$ and the edge set $E(H) \cup \{rv_1, \dots, rv_{k-\ell}\}$. Since k and ℓ are even, we have that H' is an odd graph. By Theorem 1, $\mathbf{tw}(H) \leqslant 3$. Graph H' is obtained from H by adding some vertices of degree 1, and, therefore, $\mathbf{tw}(H') \leqslant 3$. This means that when G has a connected odd subgraph H with k edges containing r, then there is a connected odd subgraph H' with k edges containing r and of treewidth at most three. But then in Step 2, we find such a graph H' with k edges.

If no connected odd subgraph with k edges was found in Step 2, then if such a graph exist, it contains no vertex of degree (in G) at least k. Therefore all such vertices can be removed from G without changing the solution. Finally, in Step 3, trying all possible connected subgraphs with k edges in the obtained graph of maximum degree at most $k-1$, we can deduce if G contains an odd subgraph with k edges.

We now analyze the running time of the algorithm. Because a connected graph with k edges has at most $k+1$ vertices, there are at most $\binom{\frac{k(k+1)}{2}}{k}$ pairwise non-isomorphic connected graphs with k edges, and we can find all connected odd graphs with k edges in time $2^{O(k \log k)}$ and check in time $O(k)$ if the treewidth of each of the graphs is at most three by making use of Bodlaender's algorithm [3]. The running time of this part can be reduced to $2^{O(k)}$, see e.g. [2]. Then for each graph H of this type, to check whether H is a subgraph of G, takes time $O(2^{O(k)} \cdot n^4 \cdot \log n)$ by Proposition 1.

When we arrive at Step 3, we have that $\Delta(G) \leqslant k-1$. For each vertex v, to check whether the graph $G[N_G^{(k)}[v]]$ induced by the vertices within distance k from v has a connected odd subgraph H with k edges, one can use a brute-force algorithm. This will take time $O(2^{O(k^2 \log k)} \cdot n)$. A smarter way of implementing Step 3, suggested to us by an anonymous STACS 2012 referee, is the following. We enumerate all connected subgraphs with $p = 0, \dots, k$ edges containing v by making use of the

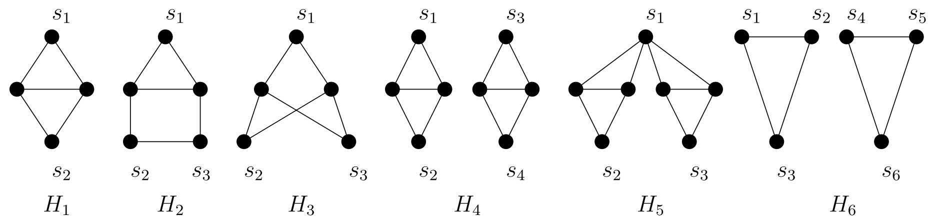

Fig. 1. The set $\mathcal{H}$.

following observation. For every connected subgraph H of G with $p \geqslant 1$ edges such that $v \in V(H)$, there is a connected subgraph H' with $p-1$ edges such that $v \in V(H')$ and H' is a subgraph of H. Hence, given all connected subgraphs with $p-1$ edges, we can enumerate all subgraphs with p edges by a brute-force algorithm.

We show by induction that for any $p \geqslant 1$, there are at most $p!(k-1)^p$ connected subgraphs with p edges in a graph G with $\Delta(G) \leqslant k-1$ that contain a given vertex v. Clearly, the claim holds for $p=1$. Let $p>1$. Any connected subgraph of G with $p-1$ edges has at most p vertices. Since there are at most $p\Delta(G) \leqslant p(k-1)$ possibilities to add an edge to this subgraph to obtain a connected subgraph with p edges, the claim follows. Therefore, for each vertex v, we can enumerate all connected subgraphs H with k edges that include v in time $O(k!(k-1)^k)$. Hence, Step 3 can be done in time $O(2^{O(k \log k)} \cdot n)$. We conclude that the total running time of the algorithm is $O(2^{O(k \log k)} \cdot n^4 \cdot \log n)$. $\square$

4. Minimal connected odd graphs with even number of edges

In this section we give the proof of Theorem 1, the main combinatorial result of this paper. It is inductive, and for the inductive step we identify specific structures in a minimal connected odd graph with an even number of edges.

To proceed with the inductive step, we need a stronger version of Theorem 1. Let G be a graph and let $x \in V(G)$. We say that a graph G' is obtained from G by *splitting x into x_1, x_2*, if G' is constructed as follows: for a partition X_1, X_2 of $N_G(x)$, we replace x by two non-adjacent vertices x_1, x_2, and join x_1, x_2 with the vertices of X_1, X_2 respectively. The following claim implies Theorem 1.

Claim 1. *Let G be a minimal connected odd graph with an even number of edges with a root r. Then* $\mathbf{tw}(G) \leqslant 3$.
Moreover, if $d_G(r)=1$ and z is the unique neighbor of r, then at least one of the following holds:

i) *there is a tree decomposition (X, T) of G of width at most three such that for any bag $X_i \in X$ with $z \in X_i$, $|X_i| \leqslant 3$; or*
ii) *for any graph G' obtained from $G-r$ by splitting z into z_1, z_2, $\mathbf{tw}(G') \leqslant 3$ and there is a tree decomposition (X, T) of G' of width at most three such that there is a bag $X_i \in X$ containing both z_1 and z_2.*

To describe the structures in the graph, we need a notion of a subgraph with terminals. Roughly speaking, a subgraph with terminals is connected to the remaining part of the graph only via terminals. More formally, let H be a subgraph of graph G, and let $s_1, \dots, s_r \in V(H)$. We say that H is a *subgraph of G with terminals $s_1, \dots, s_r$* if there is a subgraph F of G such that

- $G = F \cup H$;
- $V(F) \cap V(H) = \{s_1, \dots, s_r\}$; and
- $E(F) \cap E(H) = \emptyset$.

Thus every edge of G having at least one endpoint in a non-terminal vertex of H, should be an edge of H. In particular, terminal vertices of H separate non-terminal vertices of H from other vertices of G. Notice also that H and F are not required to be induced, i.e., if G has an edge s_is_j, then s_is_j is either in H or F. We also say that a subgraph H with a given set of terminals is *separating* if the graph obtained from G by the removal of all non-terminal vertices of H and all the edges of H (denoted $G-H$) is not connected. If it does not create confusion, we write that H is a subgraph of G or G contains H as a subgraph omitting the terminals.

The specific structures we are looking for in the inductive step are the subgraphs isomorphic to graphs with terminals from the set $\mathcal{H} = \{H_1, H_2, H_3, H_4, H_5, H_6\}$ shown in Fig. 1. Whenever we say that $H_i \in \mathcal{H}$ is contained in graph G (or G has H_i), it always means that if G has a subgraph isomorphic to H_i with the terminals shown in Fig. 1. Notice that H_6 is a subgraph of H_4 and H_5, and throughout the paper we are looking for H_6 only if we cannot find H_4 or H_5.

The proof of Claim 1 is by induction on the number of edges. As the proof is very technical, we first give a high level description.

The basis case is a graph with 6 edges. Then we assume that a minimal connected odd graph G with an even number of edges has at least 8 edges and make an inductive step.

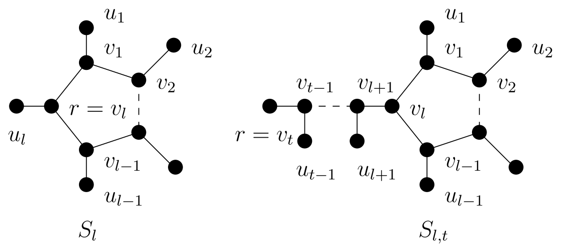

Fig. 2. Graphs S_l, $S_{l,t}$.

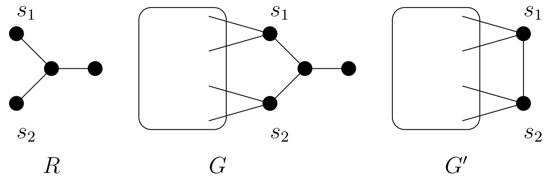

Fig. 3. Replacement of R.

If G contains a subgraph R with terminals s_1, s_2 shown in Fig. 3 such that $r \notin V(R) \setminus \{s_1, s_2\}$ and $s_1 s_2 \notin E(G)$, then the inductive step is done by replacing R by the edge $s_1 s_2$. If G has no such subgraphs, then we say that G is R-*free*, and furthermore we assume that G has this property.

The next step is to prove that if G has no subgraph from $\mathcal{H}$, then G is one of the graphs G_1, G_2, G_3 shown in Fig. 5. For each of these graphs, the theorem holds trivially. Actually, we will need a stronger result, saying that if G has no subgraph from $H_2, \ldots, H_6$ and every subgraph of G isomorphic to H_1 is of specific form, namely, this subgraph is not separating and r is not a non-terminal vertex of H_1, then even in this case, G is one of the graphs G_1, G_2, G_3 shown in Fig. 5. With this claim we can proceed further with an assumption that G contains at least one graph from $\mathcal{H}$.

For the case when r is a non-terminal vertex of a subgraph $H \in \mathcal{H}$, we prove that $H = H_1$. We remove non-terminal vertices of H, identify terminals s_1, s_2, and add a new root vertex r' adjacent to the vertex obtained from s_1, s_2. Then we prove that this graph is a minimal connected odd graph with an even number of edges, and then we can apply the induction assumption on this graph, and derive our claim for G. The difficulty here is to ensure that the treewidth of the graph G does not increase when we make the inductive step. This requires the assumptions i) and ii) in Claim 1 on the structure of tree decompositions. From this point, it can be assumed that r is not a non-terminal vertex of a subgraph from $\mathcal{H}$ with the corresponding set of terminals.

All graphs $H_2, \ldots, H_6$ have even number of edges and every terminal vertex of such a graph is of even degree. This means that G cannot contain a non-separating graph H from $\{H_2, \ldots, H_6\}$, because removing edges and non-terminal vertices of H, would result in a connected odd subgraph of G with even number of edges, which is a contradiction to the minimality of G. Hence, if G contains subgraphs from $\mathcal{H}$ but they are non-separating, G can contain only H_1. Then as we already have shown, G is one of the graphs G_1, G_2, G_3 shown in Fig. 5. Thus we can assume that G contains a separating subgraph H from $\mathcal{H}$. Among all such separating subgraphs, we select H such that the number of edges of the component F_1 of the graph $G' = G - H$ containing r is minimum. We prove that G' has exactly two components F_1, F_2, where F_1 is a tree, and this fact is used to make the inductive step.

In what follows, we give the detailed proof of Claim 1. We start with several technical lemmas.

4.1. Technical lemmas

Let $l \geqslant 3$ be an integer. The graph S_l is obtained from a cycle with l vertices by attaching a vertex of degree one to each vertex of the cycle. More formally, S_l is the graph with the vertex set $\{u_1, \ldots, u_l\} \cup \{v_1, \ldots, v_l\}$ and the edge set $\{v_1u_1, \ldots, v_lu_l\} \cup \{v_1v_l, v_1v_2, \ldots, v_{l-1}v_l\}$. For positive integers $t > l \geqslant 3$, by $S_{l,t}$ is denoted the graph obtained from a cycle of length l and a path of length $t - l$ attached to a vertex of the cycle, by attaching to every vertex of degree two a vertex of degree 1. Formally, $S_{l,t}$ has the vertex set $\{u_1, \ldots, u_{l-1}, u_{l+1}, \ldots, u_{t-1}\} \cup \{v_1, \ldots, v_t\}$ and the edge set $\{v_1u_1, \ldots, v_{l-1}u_{l-1}, v_{l+1}u_{l+1}, \ldots, v_{t-1}u_{t-1}\} \cup \{v_1v_l, v_1v_2, \ldots, v_{t-1}v_t\}$. It is assumed that S_l and $S_{l,t}$ are rooted in $r = v_l$ and $r = v_t$ respectively. The graphs S_l and $S_{l,t}$ are shown in Fig. 2. Clearly, these graphs are minimal connected odd graphs with en even number of edges rooted in r.

The following lemma describes minimal connected odd graphs with an even number of edges containing no induced cycles of length at most four.

Lemma 1. *Let G be a connected odd graph with a root r that contains cycles but has no induced cycles on three or four vertices. Then G contains either S_l for $l \geqslant 5$ or $S_{l,t}$ for $t > l \geqslant 5$ rooted in r.*

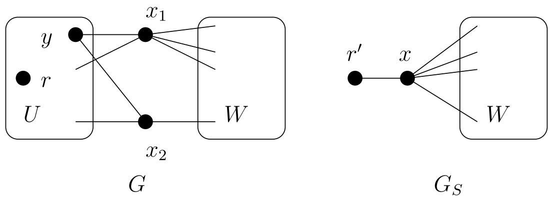

Fig. 4. Construction of G_S.

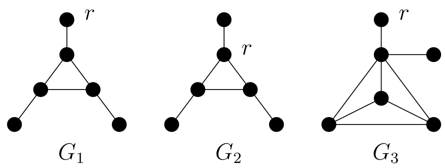

Fig. 5. Graphs G_1, G_2, G_3.

Proof. Let $C = v_1 \dots v_l$ be an induced cycle on l vertices such that $t = l + \mathrm{dist}_G(r, V(C))$ is minimum. Suppose first that $r \in V(C)$. Because G is odd, there are vertices $u_1, \dots, u_l$ adjacent to $v_1, \dots, v_l$ respectively such that $u_i \notin N_C(v_i)$ for $i \in \{1, \dots, l\}$. Since G has no C_3, C_4 and t is chosen to be minimum, $u_1, \dots, u_l$ are pairwise distinct and are distinct from $v_1, \dots, v_l$. Then we have that G contains S_l. Assume now that $r \notin V(C)$. Let $v_l \dots v_t$ where $v_t = r$ be a shortest path between $V(C)$ and r. By the same arguments, there are vertices $u_1, \dots, u_{l-1}, u_{l+1}, \dots, u_t$ adjacent to $v_1, \dots, v_{l-1}, v_{l+1}, \dots, v_{t-1}$ respectively such that $u_1, \dots, u_l$ are pairwise distinct and are distinct from $v_1, \dots, v_l$. Then we conclude that $S_{l,t}$ is a subgraph of G. $\square$

The next lemma is a straightforward observation.

Lemma 2. *Let G be a minimal connected odd graph with a root r that has an even number of edges. Suppose that G has a subgraph R shown in Fig. 3 with the terminals s_1, s_2 such that $r \notin V(R) \setminus \{s_1, s_2\}$ and $s_1 s_2 \notin E(G)$. Then the graph G' obtained by the removal of all the non-terminal vertices of R and the addition of $s_1 s_2$ is a minimal connected odd graph with the root r that has an even number of edges and $\mathbf{tw}(G) \leqslant \mathbf{tw}(G')$. Moreover, if $\mathbf{tw}(G') \geqslant 2$, then a tree decomposition of G of width at most $\mathbf{tw}(G')$ can be obtained from a tree decomposition of G' of width $\mathbf{tw}(G')$ by the addition of bags of size at most three.*

Let G be a connected odd graph with a root r and with an even number of edges. Let also $S = \{x_1, x_2\}$, $r \notin S$, be a minimal cut-set of G. Denote by U the set of vertices of the component of $G - S$ that contains r, and let $W = V(G) \setminus (S \cup U)$. Suppose that the following holds:

- each of the vertices x_1, x_2 is adjacent to exactly two vertices in $U \cup S$,
- x_1, x_2 have a common neighbor $y \in U$ such that y is not a cut-vertex,
- the number of edges of $G[W]$ be odd, and
- $(N_G(x_1) \cap N_G(x_2)) \cap W = \emptyset$.

Denote by G_S the graph obtained from $G[W]$ as follows: we add a vertex x and join it with the vertices $(N_G(x_1) \cup N_G(x_2)) \cap W$ by edges and then add a vertex r' and join it with x (see Fig. 4). The vertex r' is a root of G_S.

Lemma 3. *If G is a minimal connected odd graph with an even number of edges, then the rooted graph G_S is a minimal connected odd graph with an even number of edges.*

Proof. Clearly, G_S is a connected graph. For each $x_i \in S$, $d_G(x_i)$ is odd. Since x_i is adjacent to exactly two vertices in $U \cup S$, $|N_G(x_i) \cap W|$ is also odd. It follows that G_S an odd graph with an even number of edges, and it remains to prove that G_S is minimal. To obtain a contradiction, assume that G_S has a proper connected odd subgraph H with an even number of edges, and $r' \in V(H)$. Observe, that the edge $r'x \in E(H)$. Hence, the number of edges that join x and the vertices of W should be even. Let $X_i \subseteq W \cap N_G(x_i)$ be the set of vertices of H from $V(H) \cap W$ that are adjacent to x for $i = 1, 2$. Notice that $|X_1| + |X_2|$ is even, i.e. either $|X_1|, |X_2|$ are even or $|X_1|, |X_2|$ are odd. Consider the graph $G[U \cup S] \cup H[W \cap V(H)]$ and join x_i with the vertices of X_i for $i = 1, 2$. Denote the obtained graph by H'. The graph H' is a proper connected subgraph of G, $r \in V(H')$, and the number of edges of H' is even. If $|X_1|$ and $|X_2|$ are odd, then H' is odd, and we get a contradiction with the minimality of G. Assume that $|X_1|$ and $|X_2|$ are even. Recall that x_1, x_2 have a common neighbor $y \in U$ such that y is not a cut-vertex. We consider H'' obtained from H' by the removal of $x_1 y, x_2 y$. The graph H'' is a proper connected odd subgraph of G, $r \in V(H'')$, and the number of edges of H'' is even. We arrive at contradiction. $\square$

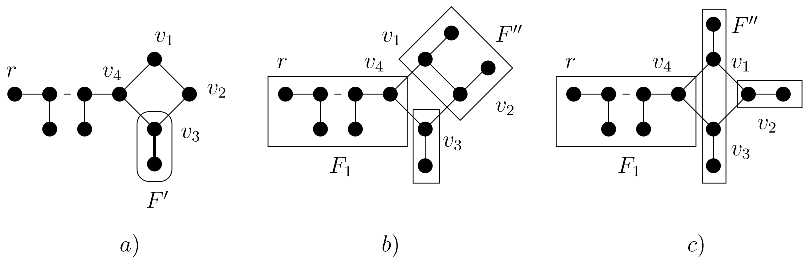

Fig. 6. Case 1.

Recall that a graph G is R-free if G has no subgraph isomorphic to the graph R with the two terminals that adjacent in G such that r is not a non-terminal vertex of this copy of R.

Lemma 4. *Let G be a minimal connected odd graph with an even number of edges rooted in r. Suppose that*

i) *G is R-free;*
ii) *G has no subgraph from $H_2, \dots, H_6$ with the corresponding terminals; and*
iii) *if G contains H_1 as a subgraph, then this subgraph is not separating and r is not a non-terminal vertex of H_1.*

Then G is one of the graphs G_1, G_2, G_3 shown in Fig. 5.

Proof. Observe that every connected odd graph with an even number of edges contains a cycle. If G has no cycles on three or four vertices, then by Lemma 1, G contains either S_l for $l \geqslant 5$ or $S_{l,t}$ for $t > l \geqslant 5$ rooted in r. By the minimality of G, either $G = S_l$ or $G = S_{l,t}$. But these graphs are not R-free; a contradiction. From now we assume that G contains induced cycles on three or four vertices. We choose a cycle C of length three of four such that the distance between $V(C)$ and r is minimum. If there are cycles of length three and four at the minimum distance from r, we assume that C is a cycle of length three.

Case 1. Suppose that $C = C_4$. We want to show that this case cannot occur. Let $C = v_1v_2v_3v_4$ and assume that $v_4 \dots v_t$ where $v_t = r$ is a shortest path between $V(C)$ and r. We consider two subcases.

Case 1.a. $r \neq v_4$. Since G is odd, we have that there are (not necessarily distinct) vertices $u_1, u_2, u_3, u_5, \dots, u_{t-1}$ adjacent to $v_1, v_2, v_3, v_5, \dots, v_{t-1}$ respectively such that $u_1 \neq v_2, v_4$ and $u_i \neq v_{i-1}, v_{i+1}$ for $i \in \{2, \dots, t-1\}$. By the choice of the cycle, $u_i \neq v_j$ for $i \in \{1, 2, 3, 5, \dots, t-1\}$ and $j \in \{1, \dots, t\}$. If it is possible to choose pairwise distinct vertices u_i, then the subgraph G' of G with $V(G') = \{u_1, u_2, u_3, u_5, \dots, u_{t-1}\} \cup \{v_1, \dots, v_t\}$ and $E(G') = \{v_1u_1, v_2u_2, v_3u_3, v_5u_5, \dots, v_{t-1}u_{t-1}\} \cup \{v_1v_4, v_1v_2, \dots, v_{t-1}v_t\}$ is an odd graph with an even number of edges, and by minimality, $G = G'$. But this graph is not R-free; a contradiction. Hence, for any choice of $u_1, u_2, u_3, u_5, \dots, u_{t-1}$, some of these vertices are same.

Observe that because C is a closest to r cycle with three or four vertices, $u_5, \dots, u_{t-1}$ are pairwise distinct. The set of edges $\{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$ is a cut-set in G because G is minimal. Let G' be the graph obtained from G by the removal of these edges. Denote by F_1 the component that contains r and v_4. Notice that F_1 should have an odd number of edges due the minimality of G.

If there is a component F' with an odd number of edges that contains a single vertex from $\{v_1, v_2, v_3\}$ (see Fig. 6a), then by the minimality of G, $F' = K_2$ and G is not R-free. This is a contradiction. If G' has four components, or G' has three components and F_1 contains two vertices from $\{v_1, v_2, v_3, v_4\}$, or G' has two components and F_1 contains three vertices from $\{v_1, v_2, v_3, v_4\}$, then such a component F' should exist. Suppose that G' has three components, and a component $F'' \neq F_1$ contains two vertices from $\{v_1, v_2, v_3\}$. Then there is a component that contains a single vertex of this set, and it should have an even number of edges. Hence, F'' has an odd number of edges. If F'' contains either v_1, v_2 or v_2, v_3 (see Fig. 6b), then since G has no H_2, it is possible to choose the vertices u_1, u_2, u_3 in such a way that they are distinct and, therefore, we have distinct $u_1, u_2, u_3, u_5, \dots, u_{t-1}$. If F'' contains v_1, v_3 (see Fig. 6c), then since G does not contain H_3, it is possible to find distinct vertices u_1, u_3 and $u_1, u_2, u_3, u_5, \dots, u_{t-1}$ would be distinct. Hence, G' has exactly two components F_1 and F_2, the numbers of edges F_1, F_2 are odd, and F_1 contains at most two vertices of the set $\{v_1, v_2, v_3, v_4\}$.

Suppose that F_1 includes two adjacent vertices from this set. Without loss of generality, let $v_1, v_4 \in V(F_1)$ and $v_2, v_3 \in V(F_2)$. Then we consider the subgraph of G obtained by the removal of the vertices of F_2 except v_2 and v_3, and the removal of the edge v_2v_3, see Fig. 7a. This graph is an odd graph with an even number of edges but this contradicts the minimality of G.

Suppose now that $v_2, v_4 \in V(F_1)$ and $v_1, v_3 \in V(F_2)$ (see Fig. 7b). Since G does not contain H_3, we can choose $u_1 \neq u_3$. Since $u_1, u_2, u_3, u_5, \dots, u_{t-1}$ are not distinct, by the choice of C, we have that $d_G(v_2) = 3$ and either $u_2 = u_6$ or $u_2 = u_5$.

Assume first that $u_2 = u_6$. There is a vertex $w \in V(F_1)$ adjacent to u_2, $w \neq v_2, v_6$. By the choice of C, $w, u_5, u_7, \dots, u_{t-1}$ are distinct. Also by the choice of C, $w \neq v_4, v_5, v_7, \dots, v_t$. Then F_1 has the subgraph with the vertex set $\{v_2, v_4, \dots, v_t, u_5,$

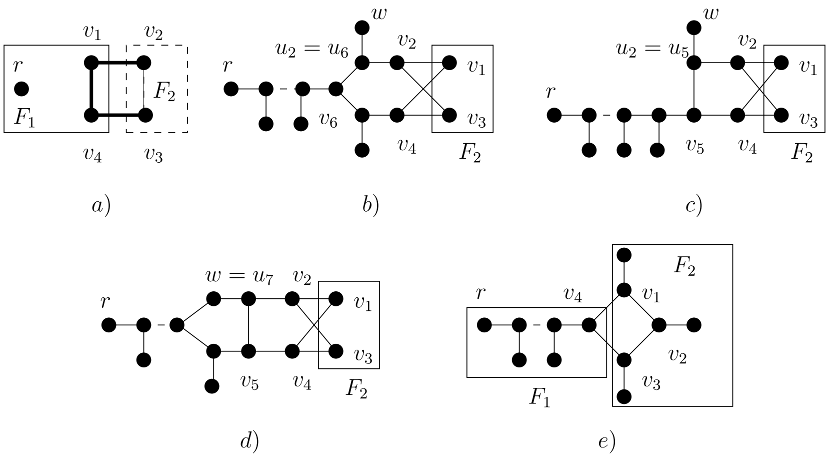

Fig. 7. Case 1.

$\dots, u_{t-1}, w\}$ and the edge set $\{v_2u_2, v_4v_5, \dots, v_{t-1}v_t,\ v_5u_5, \dots, v_{t-1}u_{t-1}, u_2w\}$, see Fig. 7b. This graph is an odd graph with an odd number of edges, and by minimality, is equal to F_1. But F_1 is not R-free, which is a contradiction.

Now, let $u_2 = u_5$. There is a vertex $w \in V(F_1)$ adjacent to u_2, $w \neq v_2, v_5$. By the choice of C, $w, u_6, u_8, \dots, u_{t-1}$ are distinct, and $w \neq v_4, v_6, \dots, v_t$. If $w \neq u_7$, then F_1 has the subgraph with the vertex set $\{v_2, v_4, \dots, v_t, u_5, \dots, u_{t-1}, w\}$ and the edge set $\{v_2u_2, v_4v_5, \dots, v_{t-1}v_t,\ v_5u_5, \dots, v_{t-1}u_{t-1}, u_2w\}$, see Fig. 7c. This graph is an odd graph with an odd number of edges, and by minimality, is equal to F_1. But F_1 is not R-free, which is a contradiction. Hence, $w = u_7$ (see Fig. 7d). Notice that $v_2u_6u_7v_7 \dots v_t$ is a shortest path between r and C. By symmetry, we can assume now that $d_G(u_4) = 3$. Let F_1' be the graph obtained from F_1 by the removal of the edges v_2u_2, v_4v_5, v_5u_5. It remains to observe that F_1' is an odd graph with even number of edges; a contradiction.

Assume now that $v_1, v_2, v_3 \in V(F_2)$ (see Fig. 7c). Because G does not contain H_2 and H_3, we again can find pairwise distinct u_1, u_2, u_3. Again, by minimality, in this case F_2 is not R-free; a contradiction.

Case 1.b. $r = v_4$. The arguments for this case are similar, and we prove that such graphs do not exist.

Observe that there are vertices u_1, u_2, u_3, u_4 adjacent to v_1, v_2, v_3, v_4 respectively, $u_i \neq v_j$ for $i, j \in \{1,2,3,4\}$. If it is possible to choose pairwise distinct vertices u_i, then the odd graph G' with $V(G') = \{u_1, \dots, u_4\} \cup \{v_1, \dots, v_4\}$ and $E(G') = \{v_1u_1, \dots, v_4u_4\} \cup \{v_1v_4, v_1v_2, \dots, v_3v_4\}$ has an even number of edges, and G' is a subgraph of G. By minimality, $G = G'$, but this graph is not R-free, and we have a contradiction. Hence, for any choice of u_1, u_2, u_3, u_4, some of these vertices are equal. The set of edges $\{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$ is a cut-set in G. Let G' be the graph obtained from G by the removal of these edges. Denote by F_1 the component that contains r and v_4. Notice that F_1 should have an odd number of edges. By the same arguments as for Case 1.a, G' has exactly two components F_1 and F_2 with odd number of edges, $v_2, v_4 \in V(F_1)$ and $v_1, v_3 \in V(F_2)$. Since G does not contain H_3, we can choose $u_1 \neq u_3$ and $u_2 \neq u_4$ and construct R in G, which is a contradiction with R-freeness.

Thus, within the assumptions of the lemma, Case 1 cannot occur.

Case 2. Let $C = v_1v_2v_3$ and assume that $v_3 \dots v_t$ where $v_t = r$ is a shortest path between $V(C)$ and r. We consider two subcases.

Case 2.a. $r \neq v_3$. Since G is odd, there are vertices $u_1, u_2, u_4, \dots, u_{t-1}$ adjacent to $v_1, v_2, v_4, \dots, v_{t-1}$ respectively such that $u_1 \neq v_2, v_3$ and $u_i \neq v_{i-1}, v_{i+1}$ for $i \in \{2, \dots, t-1\}$. By the choice of the cycle, $u_i \neq v_j$ for $i \in \{1, 2, 4, \dots, t-1\}$ and $j \in \{1, \dots, t\}$. If it is possible to choose pairwise distinct vertices u_i, then the subgraph G' of G with $V(G') = \{u_1, u_2, u_4, \dots, u_{t-1}\} \cup \{v_1, \dots, v_t\}$ and $E(G') = \{v_1u_1, v_2u_2, v_4u_4, \dots, v_{t-1}u_{t-1}\} \cup \{v_1v_4, v_1v_2, \dots, v_{t-1}v_t\}$ is an odd graph with an even number of edges, and by minimality, $G = G'$. Since G is R-free, we have that $t = 4$ and $G = G_1$. Assume that $G \neq G_1$. Then for any choice of $u_1, u_2, u_4, \dots, u_{t-1}$, some of these vertices are equal. Observe that by the choice of C, $u_4, \dots, u_{t-1}$ are pairwise distinct.

Let G' be the graph obtained from G by removing $E(C)$.

Suppose that G' is not connected. Denote by F_1 the component containing r and v_3. Notice that F_1 should have an odd number of edges. If G' has three components, then $u_1, u_2, u_4, \dots, u_{t-1}$ are pairwise distinct. Hence, G' has two components F_1, F_2, and F_2 has an even number of edges. Observe that if $u_1 = u_2$, then G contains H_1 that separates G. This contradicts condition iii) of the lemma. Therefore, we have that $u_1 = u_2$, but this is impossible, because it would imply that the u_i's are pairwise distinct. Therefore, F_1 contains two vertices of C, and we assume without loss of generality that $v_1, v_3 \in F_1$, see Fig. 8b. Then $u_2 \neq u_1, u_4, \dots, u_{t-1}$. We conclude that $u_1 = u_5$. There is a vertex $w \in V(F_1)$ adjacent to u_1, $w \neq v_2, v_5$. By the choice of C, $w, u_4, u_6, \dots, u_{t-1}$ are distinct. Then G has the subgraph with the vertex set $V(F_2) \cup \{v_1, \dots, v_t\} \cup \{u_4, \dots, u_{t-1}\} \cup \{w\}$ and the edge set $E(F_2) \cup \{v_1v_3, v_1v_2, \dots, v_{t-1}v_t\} \cup \{v_1u_1, v_4u_4, \dots, v_{t-1}u_{t-1}\} \cup \{u_1w\}$ as it is shown

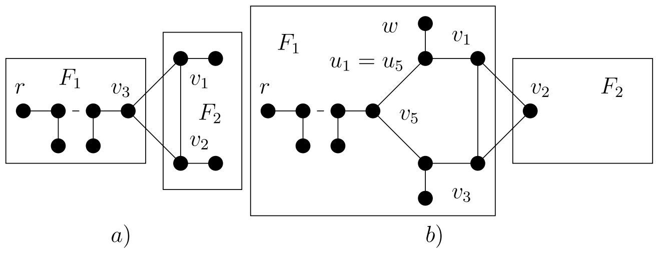

Fig. 8. Case 2, G' is not connected.

in Fig. 8b. This graph is an odd graph with an even number of edges, and by the minimality of G, G equals it, but it is not R-free and we get a contradiction. Thus we conclude that G' is connected.

If G' is a tree, then we claim that $G = G_3$.

Observe that if x_1, x_2 are leaves of G' adjacent to the same neighbor w, then $x_1 \in \{r, v_1, v_2, v_3\}$ or $x_2 \in \{r, v_1, v_2, v_3\}$. Otherwise we can delete x_1, x_2 and obtain from G a smaller odd graph with even number of edges. Notice also that if x is a leaf of G' adjacent to a vertex w of degree two in G', then $x \in \{r, v_1, v_2, v_3\}$ or $w \in \{r, v_1, v_2, v_3\}$ because G is R-free. Clearly, $|V(G')| \geqslant 4$.

Suppose that G' has exactly one inner, i.e., *non-leaf* vertex. It means that G' is a star $K_{1,l}$. Clearly, l is an odd integer and $l \geqslant 3$. If $l = 3$, then $G = K_4$, but by item iii) in lemma's assumptions, G cannot be isomorphic to K_4. Hence, $l \geqslant 5$. Then the vertices r, v_1, v_2, v_3 are leaves of G', but it contradicts the choice of C, since we have another cycle of length three at distance one from r. We conclude that G' has at least two inner vertices.

Suppose that G' has two inner vertices w_1, w_2 such that each vertex w_i is adjacent to exactly one inner vertex, and let w_1 be adjacent to at least four leaves $x_1, \dots, x_l$. At least $l - 1$ vertices of $\{x_1, \dots, x_l\}$ are in $\{r, v_1, v_2, v_3\}$ and at least one vertex of $\{x_1, \dots, x_l\}$ is adjacent to w_2. If r is adjacent to w_2, then $v_1, v_2, v_3 \in \{x_1, \dots, x_l\}$, but it contradicts the choice of C. Hence, $r \in \{x_1, \dots, x_l\}$ and exactly two vertices of $\{v_1, v_2, v_3\}$ are included in this set. Then the cycle induced by w_1 and these two vertices is at distance one from r, and it contradicts the choice of C, since C is at distance two from r in this case. We can claim now that each inner vertex w that is adjacent to one another inner vertex is adjacent to exactly two leaves of G'.

Suppose that there are three inner vertices w_1, w_2, w_3 such that each vertex w_i is adjacent to exactly one inner vertex. Let x_1^i and x_2^i be the leaves adjacent to w_i. Then at least two vertices from the set $\{w_i, x_1^i, x_2^i\}$ are included in $\{r, v_1, v_2, v_3\}$ by the minimality of G and R-freeness; a contradiction. It means that there are exactly two inner vertices w_1, w_2 such that each vertex w_i is adjacent to exactly one inner vertex. Let x_1^i and x_2^i be the leaves adjacent to w_i.

Suppose now G' has another inner vertex w. Clearly w is adjacent to l leaves $y_1, \dots, y_l$ where l is odd and $d_{G_1}(w) = l+2$. At least two vertices from each set $\{w_i, x_1^i, x_2^i\}$ are included in $\{r, v_1, v_2, v_3\}$. It follows that $w, y_1, \dots, y_l$ are not included in $\{r, v_1, v_2, v_3\}$. If $l = 1$, then we have a contradiction with the R-freeness of G, and if $l \geqslant 3$, then we have a contradiction with the minimality of G.

Now we have that G' has exactly two inner vertices and each inner vertex is adjacent to exactly two leaves. Since G does not contain H_2, we conclude that $G = G_3$.

If G' has cycles, but there are no cycles C_3, C_4, then by Lemma 1, we can find an odd subgraph of G' with an even number of edges, but it contradicts the minimality of G. From now we assume that G' contains induced cycles on three or four vertices. Let C' be a cycle on three or four such that the distance between $V(C')$ and r is minimum.

Suppose that $C' = C_4$. Let $C' = w_1w_2w_3w_4$. The set of edges $\{w_1w_2, w_2w_3, w_3w_4, w_4w_1\}$ is a cut-set in G because G is minimal, and therefore, it is a cut-set in G'. Denote by G'' the graph obtained from G' by the removal of these edges. Let F_1' be a component of G'' that contains r and assume that $w_1 \in V(F_1')$. Observe that $v_3 \in V(F_1')$ and F_1' has an odd number of edges by the minimality of G. If $v_1, v_2, v_3 \in V(F_1')$, then the graph obtained from F_1' by the addition of the edges of C is an odd subgraph of G with an even number of edges, but it contradicts the minimality of G. Hence, at least one vertex of $\{v_1, v_2\}$ is not included in F_1'. Observe also that at least one component of G'' does not contain vertices from $\{v_1, v_2, v_3\}$, since otherwise the graph obtained from G by the removal of $E(G')$ is connected, but it is impossible by the minimality of G.

First we consider the case when G'' has four components F_1', F_2', F_3', F_4', $w_i \in V(F_i')$. Assume that exactly one component F_i' has no vertices from $\{v_1, v_2, v_3\}$ (see Fig. 9a). If F_i' has an even number of edges, then the graph obtained from G by removing $V(F_i')$ and edges $w_1w_2, w_2w_3, w_3w_4, w_4w_1$ is a connected odd subgraph of G with an even number of edges, which contradicts to the minimality of G. If the number of edges of F_i' is odd, then $F_i' = K_2$ and G is not R-free or $w_{i-1}w_{i+1} \in E(C) \subseteq E(G)$ (it is assumed that $w_5 = w_1$) and the graph obtained from G by the removal of $V(F_i')$ and the edges $w_{i-1}w_i, w_iw_{i+1}, w_{i-1}w_{i+1}$ is a connected odd subgraph of G with an even number of edges; again a contradiction to minimality. Suppose that exactly two components F_i', F_j' have no vertices from $\{v_1, v_2, v_3\}$ (see Fig. 9b,c). If one of these components, say F_i', has an odd number of edges, then $F_i' = K_2$ and G is not R-free or $w_{i-1}w_{i+1} \in E(G)$ and the graph obtained from G by the removal of $V(F_i')$ and the edges $w_{i-1}w_i, w_iw_{i+1}, w_{i-1}w_{i+1}$ is a connected odd subgraph of G with an even number of edges; a contradiction. If F_i', F_j' have even numbers of edges, then the graph obtained from G by

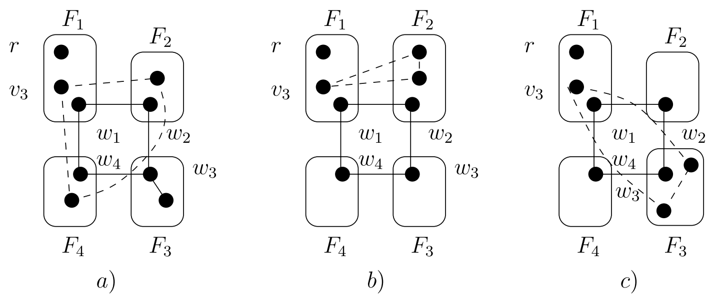

Fig. 9. The graph G'' with four components.

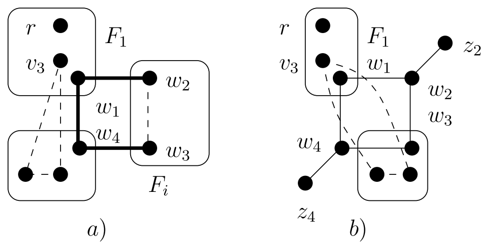

Fig. 10. The graph G'' with three components.

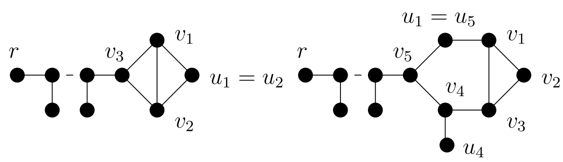

Fig. 11. The case $C = C_3$.

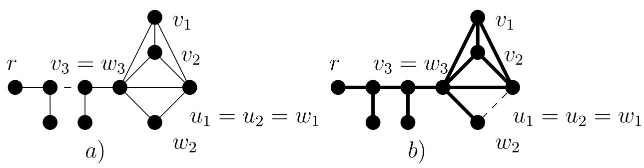

Fig. 12. The case $u_1 = u_2$.

the removal of $V(F'_i) \cup V(F'_j)$ and the edges $w_1w_2, w_2w_3, w_3w_4, w_4w_1$ is a connected odd subgraph of G with an even number of edges and we again get a contradiction to the minimality of G.

Now, let now G'' have three components F'_1, F'_2, F'_3. Then exactly one component F'_i has no vertices from $\{v_1, v_2, v_3\}$. If F'_i has an even number of edges, then the graph obtained from G by the removal of $V(F'_i)$ and the edges $w_1w_2, w_2w_3, w_3w_4, w_4w_1$ is a connected odd subgraph of G with an even number of edges and thus G is not minimal, a contradiction. Hence, the number of edges in F'_i is odd. If F'_i contains exactly one vertex $w_i \in \{w_2, w_3, w_4\}$, then by the same arguments as above, we obtain a contradiction to the minimality of G. If F'_i contains two adjacent vertices w_i, w_{i+1} (see Fig. 10a), then the graph obtained from G by the removal of $V(F'_i) \setminus \{w_i, w_{i+1}\}$ and the edge w_iw_{i+1} is a connected odd subgraph of G with an even number of edges; a contradiction. It follows that F'_i contains w_2, w_4. Since G does not contain H_3, there are $z_2, z_4 \in V(F'_i)$, $z_2 \neq z_4$, that are adjacent to w_2, w_4 respectively. Recall that the number of edges of G' is odd. Therefore, the component F'_j, $j \neq 1, i$, has an odd number of edges. Hence, the subgraph of G with the vertex set $V(F'_1) \cup V(F'_j) \cup \{w_2, w_4, z_2, z_4\}$ and the edge set $E(F'_1) \cup E(F'_j) \cup \{w_1w_2, w_2w_3, w_3w_4, w_4w_1, w_2z_2, w_4z_4\}$ (see Fig. 10b) is an odd graph with an even number of edges; a contradiction.

Suppose now that $C' = C_3$, and let $C' = w_1w_2w_3$. Observe that since G does not contain H_6, C and C' have a common vertex. Recall that it was assumed that for any choice of $u_1, u_2, u_4, \dots, u_{t-1}$, some of these vertices are equal. Observe that (up to symmetry) there are two possibilities: either $u_1 = u_2$ or $u_1 = u_5$ (see Fig. 11).

Suppose that $u_1 = u_2$. Notice that we can make another choice of these vertices unless $d_G(v_1) = d_G(v_2) = 3$, i.e., unless G contains H_1. It follows that $v_3 \in V(C')$. Assume that $v_3 = w_3$. Since G does not contains H_6, C' and the cycle $v_1v_2u_2$ should have common vertices, we can assume that $w_1 = u_1 = u_2$ (see Fig. 12a). By the choice of C, $w_2 \neq v_4, \dots, v_t$ and $w_2 \neq u_4, \dots, u_{t-1}$. Then the subgraph of G with the vertex set $\{v_1, \dots, v_t\} \cup \{u_1, u_4, \dots, u_{t-1}\} \cup \{w_2\}$ and the edge set

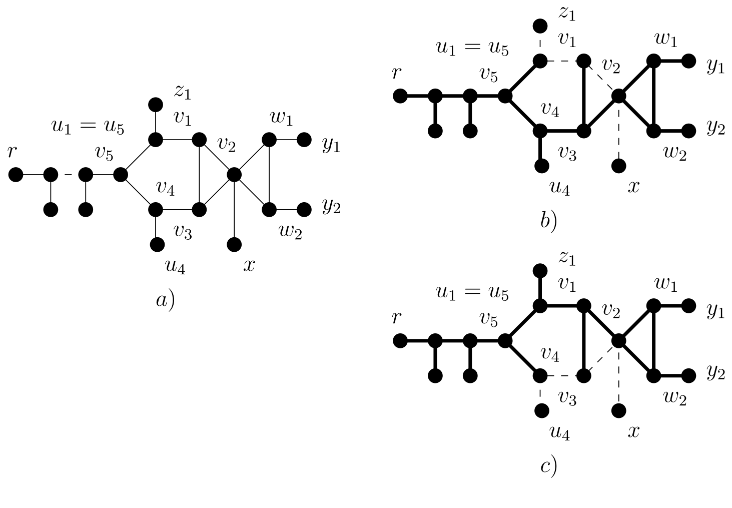

Fig. 13. The case $u_1 = u_5$.

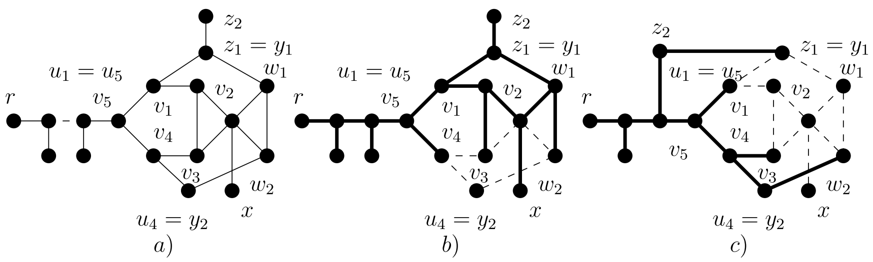

Fig. 14. The case $u_1 = u_5$.

$\{u_1v_1, u_1v_2, u_1v_3, v_1v_3, v_1v_2, \ldots, v_{t-1}v_t\} \cup \{v_4u_4, \ldots, v_{t-1}u_{t-1}\} \cup \{w_2w_3\}$ (see Fig. 12b) is a connected odd graph with an even number of edges and we have a contradiction.

Assume that $u_1 = u_5$. If $d_G(v_1) > 3$, then we can choose another vertex as u_1. Hence, we can assume that $d_G(v_1) = 3$. Notice that $v_1u_1v_5 \ldots v_t$ is a shortest path between r and C. If $d_G(v_3) > 3$, then by using this path instead of $v_3 \ldots v_t$ we obtain a contradiction. Hence, $d_G(v_1) = d_G(v_3) = 3$. Notice that $v_2 \in V(C')$ and we assume that $v_2 = w_3$. Since G is odd, there are (not necessarily distinct) vertices x, y_1, y_2, z_1 shown in Fig. 13a. Since C is a closest to r cycle with three or four vertices and G has no triangles disjoint with C or C', $x \neq z_1, u_4$, $y_1 \neq y_2$, $z_1 \neq u_4$ and $\{x, y_1, y_2, z_1\} \cap \{u_4, \ldots, u_{t-1}\} = \emptyset$. If it is possible to choose y_1, y_2 in such a way that either $u_4 \neq y_1, y_2$ or $z_1 \neq y_1, y_2$, then G has a subgraph with the vertex set $\{w_1, w_2, y_1, y_2\} \cup \{v_1, \ldots, v_t\} \cup \{u_4, \ldots, u_{t-1}\}$ and the edge set $\{w_1v_2, w_1y_1, w_1w_2, w_2v_2, w_2y_2, v_1v_3\} \cup \{v_2v_3, \ldots, v_{t-1}v_t\} \cup \{v_4u_4, \ldots, v_{t-1}u_{t-1}\}$ (Fig. 13b) or a symmetric subgraph shown in Fig. 13c, and we get a contradiction. Therefore, it can be assumed that $y_1 = z_1$, $y_2 = u_4$ and $d_G(w_1) = d_G(w_2) = 3$ (see Fig. 14a). Now we can find a vertex $z_2 \neq u_1, w_1$ adjacent to z_1. The edge set $E(C) \cup E(C')$ has an even number of edges. Hence, this is a cut-set of G. Let G' be the graph obtained from G by the removal of $E(C) \cup E(C')$. The graph G' has two components F_1, F_2, where $r, v_1, v_3, w_1, w_2 \in V(F_1)$ and $x, v_2 \in V(F_2)$. By the minimality of G, the number of edges of F_1 is odd. It follows that the number of edges of F_2 is also odd. Then we have two possibilities. It can happen that G has a connected odd subgraph with an even number of edges with the vertex set $\{v_1, v_2, w_1, z_1, u_1\} \cup \{v_5, \ldots, v_t\} \cup \{v_3, x, w_2, z_2\} \cup \{u_6, \ldots, u_{t-1}\}$ and the edge set $\{u_1v_1, v_1v_2, v_2w_1, w_1z_1, z_1u_1, u_1v_5\} \cup \{v_5v_6, \ldots, v_{t-1}v_t\} \cup \{v_1v_3, v_x, w_1w_2, z_1z_2\} \cup \{v_6u_6, \ldots, v_{t-1}u_{t-1}\}$ (see Fig. 14b). Another possibility is that the graph obtained from G by the removal of $V(F_2) \cup \{v_1, w_1\}$ and the set of edges $E(C) \cup E(C') \cup \{u_1z\}$ (see Fig. 14c) is a connected odd graph with an even number of edges. We have the first case if $z_2 \neq u_6$, and we have the second case otherwise. So, we get a contradiction.

Case 2.b. $r = v_3$. The arguments are similar to the previous case. Since G is odd, there are vertices u_1, u_2, u_3 adjacent to v_1, v_2, v_3 such that $u_i \neq v_j$ for $i, j \in \{1, 2, 3\}$. If it is possible to choose pairwise distinct vertices u_i, then the subgraph G' of G with $V(G') = \{u_1, u_2, u_3\} \cup \{v_1, v_2, v_3\}$ and $E(G') = \{v_1u_1, v_2u_2, v_3u_3\} \cup \{v_1v_3, v_1v_2, v_2v_3\}$ is an odd graph with an even number of edges, and by minimality, $G = G'$. Hence, $G = G_2$. Assume that $G \neq G_2$. Then for any choice of u_1, u_2, u_3, some of these vertices are equal.

Let G' be the graph obtained from G by the removal of $E(C)$.

Suppose that G' is not connected. Denote by F_1 the component that contains r. Notice that F_1 should have an odd number of edges. If G' has three components, then u_1, u_2, u_3 are pairwise distinct. Hence, G' has two components F_1, F_2, and F_2 has an even number of edges. If $v_1, v_2 \in V(F_2)$, then we can choose $u_1 \neq u_2$ unless G contains H_1 that separates G. Therefore, F_1 contains two vertices of C, and we assume without loss of generality that $v_1, v_3 \in F_1$. Then $u_2 \neq u_1, u_3$ and, therefore, for any choice, $u_1 = u_3$. It can happen only if $d_G(v_1) = d_G(v_3) = 3$, and it means that r is a non-terminal vertex of H_1; a contradiction.

We conclude that G' is connected.

Suppose that G' is a tree. We use the same arguments as in Case 2.a. Suppose that G' has exactly one inner vertex. It means that G' is a star $K_{1,l}$. Clearly, l is an odd integer and $l \geqslant 3$. If $l = 3$, then $G = K_4$, but by item iii) in lemma's assumptions, G cannot be isomorphic to K_4. Hence, $l \geqslant 5$. But then at least two leaves are not included in $\{v_1, v_2, v_3\}$, and we have a contradiction with the minimality of G. Hence, G' has at least two inner vertices. Suppose that G' has two inner vertices w_1, w_2 such that each vertex w_i is adjacent to exactly one inner vertex, and let w_1 be adjacent to at least two leaves $x_1^1, \dots, x_l^1$ and let w_2 be adjacent to at least two leaves $x_1^2, \dots, x_{l'}^2$. Then at least two vertices from each of the sets $\{w_1, x_1^1, \dots, x_l^1\}$ and $\{w_2, x_1^2, \dots, x_{l'}^2\}$ are included in $\{v_1, v_2, v_3\}$; a contradiction.

If G' has cycles, but there are no cycles C_3, C_4, then by Lemma 1, we can find an odd subgraph of G' with an even number of edges, but it contradicts the minimality of G. From now we assume that G' contains induced cycles on three or four vertices. Let C' be such a cycle that the distance between $V(C')$ and r is minimum.

We use exactly the same arguments as in Case 2.a and prove that $C' \neq C_4$. Suppose that $C' = C_3$, and let $C' = w_1w_2w_3$. Observe that since G does not contain H_6, C and C' have a common vertex. Recall that it was assumed that for any choice of u_1, u_2, u_3, some of these vertices are equal. Because G does not contain H_1 that has r as a non-terminal vertex, there is only one possibility: $u_1 = u_2$, and we again use the same arguments as in Case 2.a to get a contradiction. □

Now we need two technical lemmas about odd graphs when the number of edges is odd.

Lemma 5. *Let G be an odd graph with an odd number of edges rooted in r. Let also $W = \{w_1, \dots, w_p\} \subseteq V(G)$. Suppose that*

- i) *G is a minimal (not necessarily connected) odd graph with an odd number of edges such that $r \in V(G)$, $W \subseteq V(G)$, and each component of G contains at least one vertex of W;*
- ii) *G has no connected odd subgraph with an even number of edges rooted in r;*
- iii) *G is R-free;*
- iv) *G does not contain $H \in \{H_1, H_2, H_3\}$ such that r is a non-terminal vertex of H.*

Then one of the following holds:

- a) *G is a tree;*
- b) *G contains a separating $H \in \{H_1, H_2, H_3\}$ such that for a component F of $G - H$ with $r \in V(F)$, $V(F) \cap W = \emptyset$; or*
- c) *G contains an induced cycle C on three vertices such that the graph obtained from G by the removal of $E(C)$ has two components F_1, F_2 and $V(F_1) \cap W \neq \emptyset$, $V(F_2) \cap W \neq \emptyset$.*

Proof. If G has no cycles then the claim is trivial. Assume that G has cycles. If G contains no cycles on three or four vertices, then by Lemma 1, G has connected odd subgraph with an even number of edges rooted in r. Hence, G contains induced cycles on three or four vertices. Let C be such a cycle that the distance between $V(C)$ and r is minimum. We consider two cases.

Case 1. $C = C_4$. Let $C = v_1v_2v_3v_4$ and assume that $v_4 \dots v_t$ where $v_t = r$ is a shortest path between $V(C)$ and r. We consider two subcases.

Case 1.a. $r \neq v_4$. Since G is odd, there are (not necessarily distinct) vertices $u_1, u_2, u_3, u_5, \dots, u_{t-1}$ adjacent to $v_1, v_2, v_3, v_5, \dots, v_{t-1}$ respectively such that $\{u_1, u_2, u_3, u_5, \dots, u_{t-1}\} \cap \{v_1, \dots, v_t\} = \emptyset$. If it is possible to choose pairwise distinct vertices u_i, then the subgraph G' of G with $V(G') = \{u_1, u_2, u_3, u_5, \dots, u_{t-1}\} \cup \{v_1, \dots, v_t\}$ and $E(G') = \{v_1u_1, v_2u_3, v_3u_3, v_5u_5, \dots, v_{t-1}u_{t-1}\} \cup \{v_1v_4, v_1v_2, \dots, v_{t-1}v_t\}$ is a connected odd graph with an even number of edges that contain r, but it contradicts ii). Hence, for any choice of $u_1, u_2, u_3, u_5, \dots, u_{t-1}$, some of these vertices are equal.

Observe that since C is a closest to r cycle with three or four vertices, $u_5, \dots, u_{t-1}$ are pairwise distinct. The set of edges $\{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$ is a cut-set in G because G is minimal. Let G' be the graph obtained from G by the removal of these edges. Denote by F the component that contains r and v_4. Notice that F should have an odd number of edges. By condition i), $W \setminus V(F) \neq \emptyset$. Observe also that if $V(F) \cap W \neq \emptyset$, then G' has a component F' such that $V(F') \cap W = \emptyset$, since otherwise we can remove $E(G)$ and get an odd graph with an odd number of edges such that r and W are included in it, and each component contains at least one vertex of W.

If G' has four components, then we always can find distinct u_1, u_2, u_3 that are different from $u_5, \dots, u_{t-1}$. Suppose that G' has three components. Let v_4 be the unique vertex from $V(C)$ in F. Unless G contains $H \in \{H_2, H_3\}$ with a terminal in v_4 such that C is a part of H, we always can find pairwise distinct u_1, u_2, u_3. Suppose that G contains such $H \in \{H_2, H_3\}$, but the component of $G - H$ with r has at least one vertex from W. In this case G' has exactly one component F' such that $V(F') \cap W = \emptyset$. If the number of edges of F' is even, then the graph obtained from G by the removal of $V(F')$ and $E(C)$ is an odd graph with an odd number of edges that r and W are included in it, and each component contains at least one

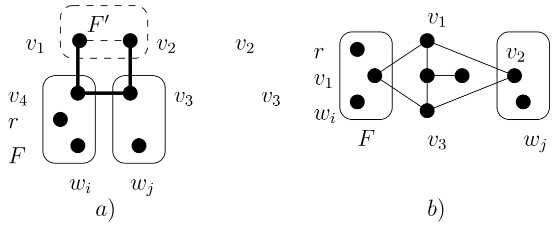

Fig. 15. The case when G' has three components.

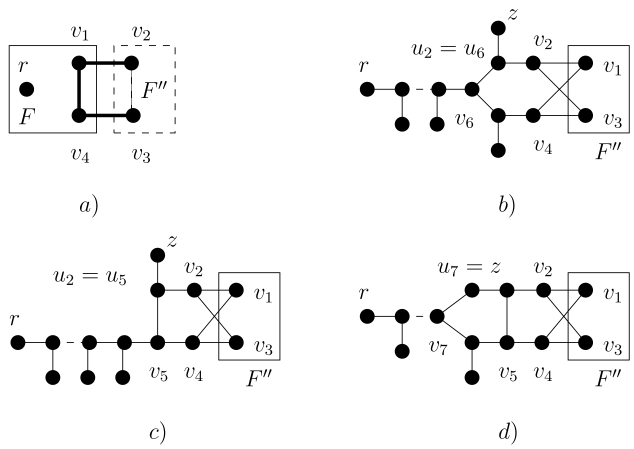

Fig. 16. The case when G' has two components.

vertex of W. It gives a contradiction with i). Hence, the number of edges of F' is odd. If F' contains exactly one vertex from $V(C)$, then by minimality, $F' = K_2$ and we get a contradiction with iii). Let F' contain two vertices from $V(C)$. Assume that F' includes two adjacent vertices of C, say the vertices v_1, v_2 (see Fig. 15a). Then the graph obtained from G by the removal of $V(F') \setminus \{v_1, v_2\}$ and the edge v_1, v_2 is a connected odd graph with an even number of edges; a contradiction. Suppose now that $v_1, v_3 \in V(F')$ (see Fig. 15b). It follows that for any choice of u_1, u_3, it should be $u_1 = u_3$, but then $H = H_3$ and $F' = R$ with the terminals v_1, v_3; a contradiction with R-freeness.

We now suppose that G' has two components F, F''. In this case $V(F) \cap W = \emptyset$ and the number of edges of F'' is even.

If F contains the unique vertex from $\{v_1, v_2, v_3, v_4\}$, then we can find pairwise distinct u_1, u_2, u_3 unless G contains $H \in \{H_2, H_3\}$ with a terminal in v_4 such that C is a part of H and b) holds. Suppose that F contains three vertices of this set. Then the component $F'' \neq F$ contains a vertex u_i adjacent to the unique vertex of v_i in $\{v_1, v_2, v_3, v_3\} \cap V(F'')$. It remains to observe that the subgraph of G with the vertex set $V(F) \cup \{v_i, u_i\}$ and the edge set $E(F) \cup E(C) \cup \{v_i u_i\}$ is a connected odd graph with an even number of edges; a contradiction. We conclude that F contains exactly two vertices of $\{v_1, v_2, v_3, v_4\}$.

Suppose that F includes two adjacent vertices from this set. Without loss of generality, let $v_1, v_4 \in V(F)$ and $v_2, v_3 \notin V(F)$ (see Fig. 16a). Then we consider the subgraph of G with the vertex set $V(F) \cup \{v_2, v_3\}$ and the edge set $E(F) \cup \{v_1 v_4, v_1 v_2, v_3 v_4\}$. This graph is an odd graph with an even number of edges, but this contradicts ii).

Suppose now that $v_2, v_4 \in V(F)$. We can find distinct u_1, u_3 adjacent to v_1, v_3 unless G contains H_3 and b) holds. Otherwise, since for any choice, not all $u_1, u_2, u_3, u_5, \ldots, u_{t-1}$ are not distinct, but $u_5, \ldots, u_{t-1}$ are pairwise distinct, we conclude that $d_G(v_2) = 3$ and either $u_2 = u_6$ or $u_2 = u_5$.

Suppose that $u_2 = u_6$ as is shown in Fig. 16b. There is a vertex $z \in V(F)$ adjacent to u_2, $z \neq v_2, v_6$. By the choice of C, $z, u_5, u_7, \ldots, u_{t-1}$ are distinct. Then F has the subgraph with the vertex set $\{v_2, v_4, \ldots, v_t\} \cup \{u_5, \ldots, u_{t-1}\} \cup \{z\}$ and the edge set $\{v_4 v_5, \ldots, v_{t-1} v_t\} \cup \{v_2 u_4, v_5 u_5, \ldots, v_{t-1} u_{t-1}\} \cup \{u_2 z\}$ as in is shown in the figure. This graph is an odd graph with an odd number of edges, and by the minimality of G, $F = G$, but it is not R-free and we get a contradiction.

Let $u_2 = u_5$. There is a vertex $z \in V(F)$ adjacent to u_2, $w \neq v_2, v_5$. By the choice of C, $z, u_6, u_8, \ldots, u_{t-1}$ are distinct, and $z \neq v_4, v_6, \ldots, v_t$. If $z \neq u_7$, then F has the subgraph with the vertex set $\{v_2, v_4, \ldots, v_t, u_5, \ldots, u_{t-1}, z\}$ and the edge set $\{v_2 u_2, v_4 v_5, \ldots, v_{t-1} v_t,\ v_5 u_5, \ldots, v_{t-1} u_{t-1}, u_2 z\}$, see Fig. 16c. This graph is an odd graph with an odd number of edges, and by minimality, is equal to F. But F is not R-free, which is a contradiction. Hence, $z = u_7$ (see Fig. 16d). Notice that $v_2 u_6 u_7 v_7 \ldots v_t$ is a shortest path between r and C. By symmetry, we can assume now that $d_G(u_4) = 3$. Let F' be the graph obtained from F by the removal of the edges $v_2 u_2, v_4 v_5, v_5 u_5$. It remains to observe that F' is an odd graph with even number of edges; a contradiction.

Case 1.b. $r = v_4$. The arguments are similar. Since G is odd, there are (not necessarily distinct) vertices u_1, u_2, u_3, u_4 adjacent to v_1, v_2, v_3, v_4 respectively such that $\{u_1, u_2, u_3, u_4\} \cap \{v_1, v_2, v_3, v_4\} = \emptyset$. If it is possible to choose pairwise distinct

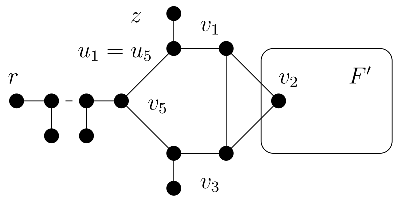

Fig. 17. The case when G' has two components.

vertices u_i, then the subgraph G' of G with $V(G') = \{u_1, u_2, u_3, u_4\} \cup \{v_1, v_2, v_3, v_4\}$ and $E(G') = \{v_1u_1, v_1u_2, v_3u_3, v_4u_4\} \cup \{v_1v_4, v_1v_2, v_2v_3, v_3v_4\}$ is an odd graph with an even number of edges that contain r, but it contradicts ii). Hence, for any choice of u_1, u_2, u_3, u_4, some of these vertices are equal. The set of edges $\{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$ is a cut-set in G because G is minimal. Let G' be the graph obtained from G by the removal of these edges. Denote by F the component that contains r. If G' has at least three components or G' has two components and F contains one or three vertices from $\{v_1, v_2, v_3, v_4\}$ or F contains two adjacent vertices from this set, then the arguments are the same as in Case 1.a. Suppose that $v_2, v_4 \in V(F)$. We can find distinct u_1, u_3 adjacent to v_1, u_3 unless G contains H_3, and we can find distinct u_2, u_4 adjacent to v_2, u_4 unless G contains H_3 such that r is a non-terminal vertex of H_3. If G contains H_3, then b) holds, and the second case is impossible because of iii).

Case 2. $C = C_3$. Let $C = v_1v_2v_3$ and assume that $v_3 \dots v_t$ where $v_t = r$ is a shortest path between $V(C)$ and r. We consider two subcases.

Case 2.a. $r \neq v_3$. Since G is odd, there are (not necessarily distinct) vertices $u_1, u_2, u_4, \dots, u_{t-1}$ adjacent to $v_1, v_2, v_4, \dots, v_{t-1}$ respectively such that $\{u_1, u_2, u_4, \dots, u_{t-1}\} \cap \{v_1, \dots, v_t\} = \emptyset$. If it is possible to choose pairwise distinct vertices u_i, then the subgraph G' of G with $V(G') = \{u_1, u_2, u_4 \dots, u_{t-1}\} \cup \{v_1, \dots, v_t\}$ and $E(G') = \{v_1u_1, v_2u_2, v_4u_4, \dots, v_{t-1}u_{t-1}\} \cup \{v_1v_4, v_1v_2, \dots, v_{t-1}v_t\}$ is an odd graph with an even number of edges that contain r, but it contradicts ii). Hence, for any choice of $u_1, u_2, u_4, \dots, u_{t-1}$, some of these vertices are equal.

Observe that since C is a closest to r cycle with tree or four vertices, $u_4, \dots, u_{t-1}$ are pairwise distinct. Let G' be the graph obtained from G by the removal of the set of edges $\{v_1v_2, v_2v_3, v_3v_1\}$. The graph G' is an odd subgraph with an even number of edges. Hence, G' is not connected. Denote by F the component that contains r and v_4. Notice that F should have an odd number of edges.

If G' has three components, then we always can find distinct u_1, u_2, u_3 that are different from $u_5, \dots, u_{t-1}$. Suppose that G' has two components F, F'.

If $V(F) \cap W \neq \emptyset$ and $V(F') \cap W \neq \emptyset$, then c) holds. If $V(F') \cap W = \emptyset$, then F is a connected odd graph with an odd number of edges such that $r \in V(F)$, $W \subseteq V(F)$, and we get a contradiction with i). Assume that $V(F) \cap W = \emptyset$.

If F contains the unique vertex from $\{v_1, v_2, v_3\}$, then either G contains separating H_1 or we can find pairwise distinct u_1, u_2. Let F contain two vertices of this set. Without loss of generality we assume that $v_1, v_3 \in V(F)$. Clearly, $u_2 \neq u_1, u_4 \dots, u_{t-1}$. Since for any choice, not all $u_1, u_2, u_3, u_5, \dots, u_{t-1}$ are distinct, we conclude that $u_1 = u_5$ (see Fig. 17). There is a vertex $z \in V(F)$ adjacent to u_1, $z \neq v_1, v_5$. By the choice of C, $z, u_4, u_6, \dots, u_{t-1}$ are distinct and $z \neq v_2, \dots, v_t$. Then F has the subgraph with the vertex set $\{v_1, v_3, \dots, v_t\} \cup \{u_4, \dots, u_{t-1}\} \cup \{z\}$ and the edge set $\{v_3v_4 \dots, v_{t-1}v_t\} \cup \{v_1u_1, v_4u_4, \dots, v_{t-1}u_{t-1}\} \cup \{u_1z\}$ as in is shown in Fig. 17. This graph is an odd graph with an odd number of edges, and by the minimality of G, F equals it, but it is not R-free and we get a contradiction.

Case 2.b. $r = v_3$. The arguments are similar. Since G is odd, there are (not necessarily distinct) vertices u_1, u_2, u_3 adjacent to v_1, v_2, v_3 respectively such that $\{u_1, u_2, u_3\} \cap \{v_1, v_2, v_3\} = \emptyset$. If it is possible to choose pairwise distinct vertices u_i, then the subgraph G' of G with $V(G') = \{u_1, u_2, u_3\} \cup \{v_1, v_2, v_3\}$ and $E(G') = \{v_1u_1, v_1u_2, v_3u_3\} \cup \{v_1v_3, v_1v_2, v_2v_3\}$ is an odd graph with an even number of edges that contain r, but it contradicts ii). Hence, for any choice of u_1, u_2, u_3, some of these vertices are equal. As in Case 2.a, the set of edges $\{v_1v_2, v_2v_3, v_3v_1\}$ is a cut-set in G. Let G' be the graph obtained from G by the removal of these edges. Denote by F the component that contains r and v_3. Notice that F should have an odd number of edges.

If G' has three components or if G' has two components F, F' such that $V(F) \cap W \neq \emptyset$ and $V(F') \cap W \neq \emptyset$, then we argue exactly as in Case 2.a. Then we can assume that $V(F) \cap W = \emptyset$.

If F contains the unique vertex from $\{v_1, v_2, v_3\}$, then either G has a separating H_1 and b) holds, or we can find pairwise distinct u_1, u_2. Suppose that F contains two vertices of this set. Without loss of generality we assume that $v_1, v_3 \in V(F)$. But then we can find distinct v_1, v_3 unless if G contains H_1 such that r is not a non-terminal vertex of H_1, and we get a contradiction with iii). $\square$

Lemma 6. *Let G be a connected odd graph with an odd number of edges, $r_1, r_2 \in V(G)$. Suppose that*

i) *G is a minimal connected odd graph with an odd number of edges that contains r_1, r_2;*

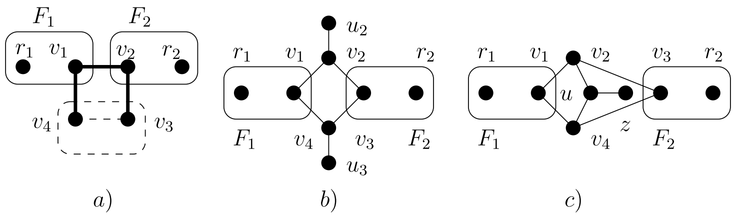

Fig. 18. The case $C = C_4$.

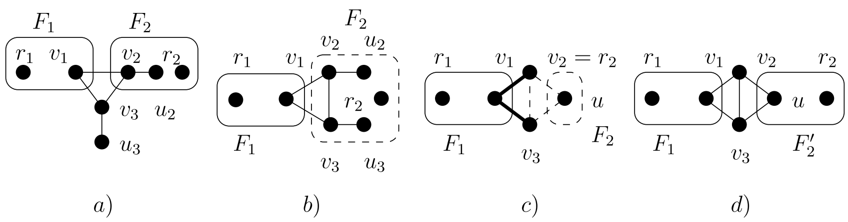

Fig. 19. The case $C = C_3$.

ii) *G is R-free;*
iii) *G has no connected odd subgraph with an even number of edges rooted in r_1 or in r_2.*

Then $\mathbf{tw}(G) \leqslant 2$.

Proof. We prove the lemma by the induction on the number of edges. For the base case, when there is only one edge, $G = K_2$ and the claim holds trivially. If G is a tree, then $\mathbf{tw}(G) = 1$. Assume that G contains cycles. If G has no cycles on three or four vertices, then by Lemma 1, we get a contradiction with iii). We consider two cases.

Case 1. The graph G contains an induced C_4.

Let $C = v_1v_2v_3v_4$ be an induced cycle. The set of edges $\{v_1v_2, v_2v_3, v_3v_4, v_1v_4\}$ is a cut-set in G. Let G' be the graph obtained from G by the removal of these edges. Denote by F_1 the component that contains r_1 and assume that $v_1 \in V(F_1)$. Observe that F_1 should have an odd number of edges. By condition i), $r_2 \notin V(F_1)$. Denote by F_2 the component that contains r_2. By iii), F_2 has an odd number of edges. Therefore, G' has at least three components and the total number of edges in the remaining components is odd.

Suppose that $|V(F_1) \cap V(C)| = 1$ and $|V(F_2) \cap V(C)| = 1$. Let F_2 contain a vertex of C adjacent with v_1, say the vertex v_2. Then the subgraph obtained from G by the removal of the vertices $V(G) \setminus (V(F_1) \cup V(F_2) \cup \{v_3, v_4\})$ and the edge v_3v_4 (see Fig. 18a) is a connected odd graph with an odd number of edges that contain r_1, r_2, and we get a contradiction with i). Suppose that $v_3 \in V(F_2)$. If there are distinct vertices $u_2, u_4 \neq v_1, v_4$ adjacent to v_2, v_4 respectively, then the graph with the vertex set $V(F_1) \cup V(F_2) \cup \{v_2, v_4, u_2, u_4\}$ and the edge set $E(F_1) \cup E(F_2) \cup E(C) \cup \{v_2u_2, v_4u_4\}$ (see Fig. 18b) is a connected odd graph with an even number of edges that contains r_1, and we get a contradiction with iii). Otherwise, $d_G(v_2) = d_G(v_4) = 3$ and there is the unique vertex $u \neq v_1, v_3$ adjacent to v_2, v_4. Also there is a vertex $z \neq v_2, v_4$ adjacent to u. The graph with the vertex set $V(F_1) \cup V(F_2) \cup \{v_2, v_4, u, z\}$ and the edge set $E(F_1) \cup E(F_2) \cup E(C) \cup \{v_2u, v_4u, uz\}$ is a connected odd graph with an odd number of edges that contains r_1, r_2 (see Fig. 18c). By minimality, this graph equals G, but then G is not R-free; a contradiction.

Suppose now that either $V(F_1) \cap V(C) = 2$ or $V(F_2) \cap V(C) = 2$. Then there is the unique component $F_3 \neq F_1, F_2$ that contains exactly one vertex $v_i \in \{v_2, v_3, v_4\}$, and this components contains an odd number of edges. By the minimality of G, $F_3 = K_2$, but then G is not R-free and we get a contradiction with ii).

We conclude that G cannot contain induced cycles on four vertices.

Case 2. The graph G contains a cycle $C = v_1v_2v_3$. Let G' be the graph obtained from G by the removal of the set of edges $\{v_1v_2, v_2v_3, v_3v_1\}$. The graph G' is an odd graph with an even number of edges that contains r_1. Hence, G' is not connected. Denote by F_1 the component that contains r_1 and assume that $v_1 \in V(F_1)$. Observe that F_1 should have an odd number of edges and by i), $r_2 \notin V(F_1)$. Let F_2 be the component of G' that contain r_2 and assume that $v_2 \in V(F_2)$. If there is the third component F_3, then $v_3 \in V(F_3)$, and F_3 has a vertex u_3 adjacent to v_3. Since F_2 contains a vertex u_2 adjacent to v_2, the subgraphs of G with the vertex set $V(F_1) \cup \{v_2, v_3, u_2, u_3\}$ and the edge set $E(F_1) \cup E(C) \cup \{v_2u_2, v_3u_3\}$ (see Fig. 19a) is a connected odd graph with an even number of edges that contains r_1, and we get a contradiction with iii). Hence, either $v_3 \in V(F_1)$ or $v_3 \in V(F_2)$. Without loss of generality, we assume that $v_3 \in V(F_2)$.

The graph F_2 contains two (not necessarily distinct) vertices u_2, u_3 adjacent to v_2, v_3 respectively. If it is possible to choose distinct u_2, u_3, then the subgraphs of G with the vertex set $V(F_1) \cup \{v_2, v_3, u_2, u_3\}$ and the edge set $E(F_1) \cup E(C) \cup \{v_2u_2, v_3u_3\}$ (see Fig. 19b) is a connected odd graph with an even number of edges that contains r_1, and we get a

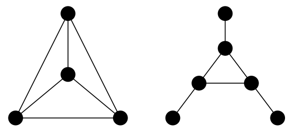

Fig. 20. The base of the induction: Minimal graphs with six edges.

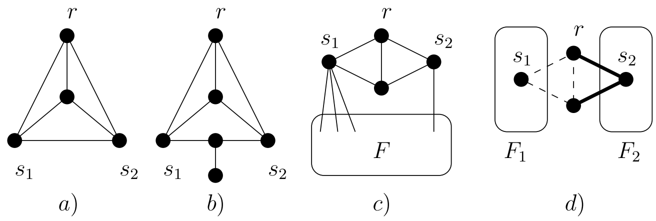

Fig. 21. Case 2, $H = H_1$

contradiction with iii). Hence, $d_G(v_2) = d_G(v_3) = 3$ and F_2 has the unique vertex u adjacent to v_2, v_3. If $r_2 = v_2$ or v_3, then the subgraphs of G with the vertex set $V(F_1) \cup \{v_2, v_3\}$ and the edge set $E(F_1) \cup \{v_1v_2, v_1v_3\}$ (see Fig. 19c) is a connected odd graph with an odd number of edges that contains r_1, r_2, and we get a contradiction with i). It means that $r_2 \neq v_2, v_3$. Denote by F'_2 the graph obtained from F_2 by the removal of v_2, v_3 (see Fig. 19d).

We apply the lemma inductively for F_1 with the vertices r_1, v_1 and F'_2 with the vertices u, r_2. These graphs have tree decompositions $(T^{(1)}, X^{(1)})$ and $(T^{(2)}, X^{(2)})$ respectively of width at most two. Let i be a node of $T^{(1)}$ with $v_1 \in X^{(1)}_i$ where $X^{(1)}_i$ is the bag of $(T^{(1)}, X^{(1)})$ which corresponds to i. Let also j be a node of $T^{(2)}$ with $u \in X^{(2)}_j$. We construct a tree decomposition for G as follows. We consider the trees $T^{(1)}$ and $T^{(2)}$ and join the nodes by a path of length three with the nodes i, i', j', j and let the bags $X_{i'} = \{v_1, v_2, v_2\}$, $X_{j'} = \{v_2, v_2, u\}$ for the nodes i', j'. The bags for other nodes are the same as in $(T^{(1)}, X^{(1)})$ and $(T^{(2)}, X^{(2)})$. It remains to observe that we get a tree decomposition for G of width at most two. $\square$

4.2. Induction

In this section we prove our inductive Claim 1 that yields Theorem 1.

Let G be a graph and let $x \in V(G)$. Recall that we say that a graph G' is obtained from G by a *splitting of x into x_1, x_2*, if G' is constructed as follows: for a partition X_1, X_2 of $N_G(x)$, we replace x by two vertices x_1, x_2, and join x_1, x_2 with the vertices of X_1, X_2 respectively.

We prove the claim by the induction on the number of edges.

Observe that every connected odd graph with an even number of edges has at least 6 edges, and there are two graphs shown in Fig. 20 with 6 edges that have these properties. It is straightforward to check that the claim holds for these graphs for any choice of the root vertex. Assume now that G has at least 8 edges.

If G is not R-free, i.e., it contains a subgraph R with the terminals s_1, s_2 such that $r \notin V(R) \setminus \{s_1, s_2\}$ and $s_1s_2 \notin E(G)$, then by Lemma 2, the graph G' obtained by the removal of all the non-terminal vertices of R and the addition of s_1s_2 is minimal and $\mathbf{tw}(G) \leqslant \mathbf{tw}(G')$. Since $|E(G')| < |E(G)|$, our claim holds by induction. From now we assume that G is R-free.

We proceed with the following case analysis.

Case 1. The graph G has no subgraphs $H_2, \ldots, H_6$, and if G contains H_1, then this subgraph is not separating and r is not a non-terminal vertex of H_1. Then by Lemma 4, G is one of the graphs G_1, G_2, G_3 shown in Fig. 5 and the claim holds.

Case 2. The graph G contains a subgraph $H \in \mathcal{H}$ and r is a non-terminal vertex of H. Clearly, we can assume that $H = H_1, H_2$ or H_3. Indeed, all vertices of H_6 are terminals, and H_4 and H_5 contain H_1. Let F be the subgraph of G obtained by the removal of the non-terminal vertices of H and all the edges of H.

Case 2.1. $H = H_1$. We have two subcases here.

Case 2.1.a. F is connected. If s_1 and s_2 are adjacent in G, then, by minimality, G is the graph shown if Fig. 21a, but this graph has six edges. If s_1 and s_2 have a common neighbor $x \in V(F)$, then since G is minimal and odd, G is the graph shown in Fig. 21b, but it contradicts our assumption that G is R-free. Let $N_G(s_1) \cap N_G(s_2) \cap V(F) = \emptyset$ (see Fig. 21c). Consider the graph G_S for $S = \{s_1, s_2\}$ with the root r' and the unique vertex x adjacent to r' (see Fig. 4; $x_1 = s_1$, $x_2 = s_2$ and $W = V(F)$ here). By Lemma 3, G_S is a minimal connected odd graph with an even number of edges. Because $|E(G_S)| \leqslant |E(G)| - 4$, by the induction hypothesis, $\mathbf{tw}(G_S) \leqslant 3$ and conditions i) or ii) of Claim 1 hold for G_S.

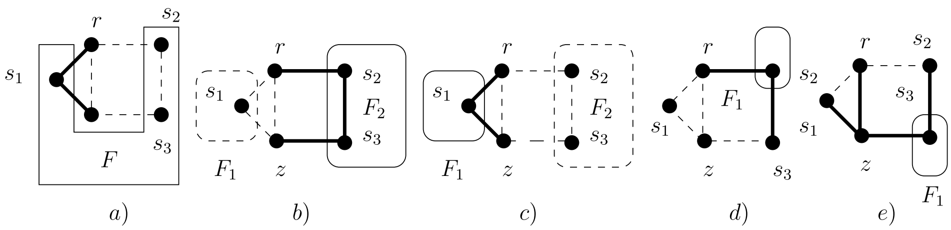

Fig. 22. Case 2, $H = H_2$.

We construct the tree decomposition of G as follows. If there is a tree decomposition (X, T) of G of width at most three such that for any bag $X_i \in X$ with $x \in X_i$, $|X_i| \leqslant 3$, then we remove r' from all the bags, replace x by s_1, s_2 in every bag, and finally, choose a node of T with the bag X_i that contain s_1, s_2, add a leaf j adjacent to this node, and let $X_j = V(H_1)$. If for any graph G' obtained from $G_S - r'$ by splitting x into x_1, x_2, $\mathbf{tw}(G') \leqslant 3$ and there is a tree decomposition (X, T) of G' of width at most three such that there is a bag $X_i \in X$ with $x_1, x_2 \in X_i$, then there is a tree decomposition (X, T) of F of width at most three such that there is a bag $X_i \in X$ with $s_1, s_2 \in X_i$. We add a leaf j adjacent to i in T, and let $X_j = V(H_1)$. It is straightforward to check that we get a tree decomposition of width at most three.

Case 2.1.b. F is not connected. Let F_1 and F_2 be components of F such that $s_1 \in V(F_1)$ and $s_2 \in V(F_2)$. Since the number of edges of G is even, by symmetry, without loss of generality, we assume that $|E(F_1)|$ is odd and $|E(F_2)|$ is even. Then the graph obtained by the removal of the vertices F_1 and the edge between non-terminal vertices of H_1 (see Fig. 21d) is a connected odd graph with an even number of edges, which contradicts the minimality of G.

Case 2.2. $H = H_2$. We prove that this case cannot occur. We consider three subcases.

Case 2.2.a. F is connected. Then the graph obtained by the removal of the edges of the cycle with four vertices induced by the non-terminal vertices of H_5 and s_2, s_3 (see Fig. 22a) is a connected odd graph with an even number of edges, and it contradicts the minimality of G.

Case 2.2.b. F has two components F_1, F_2. Since the number of edges of G is even, either both the graphs F_1, F_2 have odd numbers of edges, or they have even numbers of edges. Observe that one of these graphs, say F_1, contains exactly one vertex of the set $\{s_1, s_2, s_3\}$. Assume that the number of edges of F_1 is odd. If $s_1 \in V(F_1)$, then the graph obtained by the removal of the vertices F_1 and the edge between non-terminal vertices of H_2 is a connected odd graph with an even number of edges, and it contradicts the minimality of G (see Fig. 22b). If $s_2 \in V(F_1)$ or $s_3 \in V(F_1)$, then by the minimality of G, $F_1 = K_2$ and the graph G is not R-free, but it contradicts our assumption. Suppose now that the number of edges of F_1 is even. If $s_1 \in V(F_1)$, then the graph obtained by the removal of the vertices F_2 and the edge between non-terminal vertices of H_1 (see Fig. 22c) is a connected odd graph with an even number of edges, and it contradicts the minimality of G. Assume that $rs_2 \in E(G)$ and denote by z the second non-terminal vertex of H_2. If $s_2 \in V(F_1)$, then the subgraph of G induced by $V(F_1) \cup \{r, s_3\}$ is a connected odd graph with an even number of edges (see Fig. 22d), but this is impossible. If $s_3 \in V(F_1)$, then the subgraph of G obtained by the removal of all vertices of $V(F_1) - \{s_1, s_2\}$ (see Fig. 22e), the edges of F_1, and the edges s_1r, rs_2 is a connected odd graph with an even number of edges; a contradiction.

Case 2.2.c. F has three components F_1, F_2, F_3. Since the number of edges of G is even, one of these graphs, say F_1, contains an even number of edges. By using exactly the same arguments as in Case 2.2.b, we get a contradiction with the minimality of G.

Case 2.3. $H = H_3$. To show that this case cannot occur, we consider three subcases.

Case 2.3.a. F is connected. Denote by z the second non-terminal vertex of H_3. Then the graph obtained by the removal of the edges of the cycle with four vertices induced by r, s_2, z, s_3 is a connected odd graph with an even number of edges, and it contradicts the minimality of G.

Case 2.3.b. F has two components F_1, F_2. Since the number of edges of G is even, either both the graphs F_1, F_2 have odd numbers of edges, or they have even numbers of edges. Observe that one of these graphs, say F_1, contains exactly one vertex of the set $\{s_1, s_2, s_3\}$. Let $s_1 \in V(F_1)$. If the number of edges of F_1 is odd, then by the minimality of G, $F_1 = K_2$ and the graph G is not R-free, but it contradicts our assumption. If the number of edges of F_1 is even, then the graph obtained from G by the removal of the vertices of F_2 is a connected odd graph with an even number of edges, and it contradicts the minimality of G.

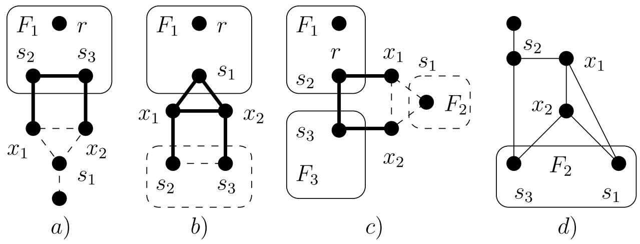

Fig. 23. The case $H = H_2$.

Case 2.3.c. F has three components F_1, F_2, F_3. Since the number of edges of G is even, one of these graphs, say F_1, contains an even number of edges. By using exactly the same arguments as in Case 2.3.b, we get a contradiction with the minimality of G.

From now we assume that the root r is not a non-terminal vertex of a graph from $\mathcal{H}$. If G would contain a non-separating subgraph H_1, then as we already have shown in Case 1, G is one of the graphs G_1, G_2, G_3 shown in Fig. 5. All graphs from $H_2, \ldots, H_6$ have even number of edges and every terminal vertex of such a graph is of even degree. This means, that G cannot contain a non-separating graph H from $H_2, \ldots, H_6$, because removing edges and non-terminal vertices of H, would result in a connected odd subgraph of G with even number of edges, which is a contradiction to the minimality of G.

Case 3. The graph G has a separating subgraph from $\mathcal{H}$.

We choose $H \in \mathcal{H}$ in such a way that the number of edges of the component F_1 with the root vertex r of $G' = G - H$ is minimum. By the minimality of G, the number of edges of F_1 is odd. Now we consider the following cases.

Case 3.a. $H = H_1$. Assume that $s_1 \in V(F_1)$ and let F_2 be the second components of G', $s_2 \in V(F_2)$. Since G has an even number of edges, the number of edges of F_2 is even. We apply Lemma 5 for $W = \{s_1\}$. By the choice of H, F_1 cannot contain a separating $H' \in \{H_1, H_2, H_3\}$ such that for a component F' of $F_1 - H'$ with $r \in V(F')$, $V(F') \cap W = \emptyset$. Since $|W| = 1$, G cannot contain an induced cycle C on three vertices such that the graph obtained from F_1 by the removal of $E(C)$ has two components F_1', F_2' and $V(F_1') \cap W \neq \emptyset$, $V(F_2') \cap W \neq \emptyset$. It follows that F_1 is a tree, and by minimality, $F_1 = K_2$.

By the minimality of G, F_2 is a minimal connected odd graph with an even number of edges with a root s_2. By induction, $\mathbf{tw}(F') \leqslant 3$. Let (X, T) be a tree decomposition of F' of width at most three, and assume that $s_2 \in X_i$. Denote by x_1, x_2 the non-terminal vertices of H. We add a path of length three with the nodes i, i_1, i_2, i_3 to T, and set $X_{i_1} = \{s_2, x_1, x_2\}$, $X_{i_2} = \{s_1, x_1, x_2\}$, and $X_{i_3} = V(F_1)$. We get a tree decomposition of G of width at most three. Notice that if $d_G(r) = 1$, then s_1 is adjacent to r, and s_1 is included in two bags of size at most three.

Case 3.b. $H = H_2$. Denote by x_1, x_2 the non-terminal vertices of H_5 adjacent to s_2, s_3 respectively.

Suppose that F_1 contains two terminal vertices of H_2. Then G' has two components, and we denote by F_2 the second component. The graph F_2 has an odd number of edges, and by minimality, $F_2 = K_2$. Since G is R-free, $s_2, s_3 \in F_1$. But then the subgraph of G with the vertex set $V(F_1) \cup \{x_1, x_2\}$ and the edge set $E(F_1) \cup \{s_2x_1, s_3x_2\}$ (see Fig. 23a) is a connected odd graph with an even number of edges; a contradiction. Hence, F_1 contains exactly one vertex of H_2. If $s_1 \in V(F_1)$, then the subgraph of G with the vertex set $V(F_1) \cup V(H)$ and the edge set $E(F_1) \cup \{s_1x_1, s_1x_2, x_1x_2, x_1s_2, x_2s_3\}$ (see Fig. 23b) is a connected odd graph with an even number of edges; a contradiction. Therefore, we can assume that $s_2 \in V(F_1)$ and $s_1, s_3 \notin V(F_1)$.

Suppose that s_1, s_3 are vertices of different components F_2, F_3 of G' respectively. If the number of vertices F_3 is odd, then $F_3 = K_2$ and we get a contradiction with R-freeness. Hence, the number of edges of F_3 is even and the number of edges of F_2 is odd. Then the graph obtained from G by the removal of $V(F_2)$ and the edge x_1x_2 (see Fig. 23c) is a connected odd graph with an even number of edges; a contradiction. It follows that s_1, s_3 are vertices of one component of G'. We denote it by F_2. Notice that the number of edges of F_2 is odd.

By exactly the same arguments as in Case 3.a, we claim that $F_1 = K_2$ (see Fig. 23d).

If s_1, s_3 are adjacent then $F_2 = K_2$ and the claim of the lemma holds. If s_1, s_3 have a common neighbor u in F_2, then there is a vertex $w \in V(F_2)$, $w \neq s_1, s_3$, adjacent to u. By minimality, $V(F_2) = \{s_1, s_3, u, w\}$ and $E(F_2) = \{s_1u, s_3u, uw\}$, but then G is not R-free; a contradiction. Otherwise we consider the graph G_S for $S = \{s_1, s_3\}$ with the root r' and the unique vertex x adjacent to r'. By Lemma 3, G_S is a minimal connected odd graph with an even number of edges. By the induction, $\mathbf{tw}(G_S) \leqslant 3$ and either i) or ii) holds for G_S

Suppose that G_S has a tree decomposition (X, T) such that for any bag $X_i \in X$, if $x \in X_i$, then $|X_i| \leqslant 3$. We construct the tree decomposition of G as follows. First we remove r' from all the bags and replace x by s_1, s_3 in every bag. Let i be a node of T with the bag X_i that contain s_1, s_3. We add a path of length three with the nodes i, i_1, i_2, i_3 to T, and set $X_{i_1} = \{s_1, s_3, x_1, x_2\}$, $X_{i_2} = \{s_2, s_3, x_1\}$, and $X_{i_3} = V(F_1)$. We get a tree decomposition of G of width at most four. Notice that if $d_G(r) = 1$, then s_2 is adjacent to r, and s_2 is included in two bags of size at most three.

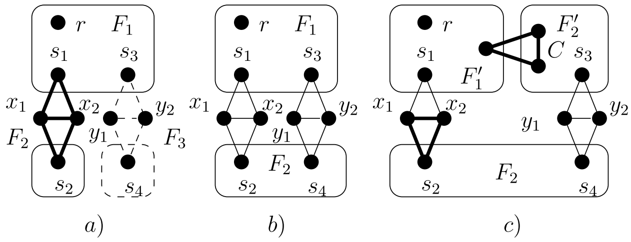

Fig. 24. The case $H = H_4$.

Suppose now that for any graph G' obtained from $G_S - r'$ by splitting x into z_1, z_2, $\mathbf{tw}(G') \leqslant 3$ and there is a tree decomposition (X, T) of G' of width at most three such that there is a bag $X_i \in X$ with $z_1, z_2 \in X_i$. Then there is a tree decomposition (X, T) of F_2 of width at most three such that there is a bag $X_i \in X$ with $s_1, s_3 \in X_i$. We add a path of length three with the nodes i, i_1, i_2, i_3 to T, and set $X_{i_1} = \{s_1, s_3, x_1, x_2\}$, $X_{i_2} = \{s_2, s_3, x_1\}$, and $X_{i_3} = V(F_1)$. We get a tree decomposition of G of width at most four. Notice that if $d_G(r) = 1$, then s_2 is adjacent to r, and s_2 is included in two bags of size at most three.

Case 3.c. $H = H_3$. Denote by x_1, x_2 the non-terminal vertices of H_3.

Suppose that F_1 contains two terminal vertices of H_3. Then G' has two components, and we denote by F_2 the second component. The graph F_2 has an odd number of edges, and by minimality, $F_2 = K_2$; a contradiction to the assumption that the graphs are R-free. Therefore, we can assume that $s_1 \in V(F_1)$ and $s_2, s_3 \notin V(F_1)$. Suppose that s_2, s_3 are vertices of different components F_2, F_3 of G' respectively. Then either F_2 or F_3 has an odd number of edges and we again get a contradiction with R-freeness. It follows that s_2, s_3 are vertices of one component of G'. We denote it by F_2. Notice that the number of edges of F_2 is odd.

By exactly the same arguments as in Case 3.a, $F_1 = K_2$.

If s_2, s_3 are adjacent then $F_2 = K_2$ and the claim of the lemma holds. If s_2, s_3 have a common neighbor u in F_2, then there is a vertex $w \in V(F_2)$, $w \neq s_2, s_3$, adjacent to u. By minimality, $V(F_2) = \{s_2, s_3, u, w\}$ and $E(F_2) = \{s_2u, s_3u, uw\}$, but then G is not R-free; a contradiction. Otherwise we consider the graph G_S for $S = \{s_2, s_3\}$ with the root r' and the unique vertex x adjacent to r'. By Lemma 3, G_S is a minimal connected odd graph with an even number of edges. By the induction, $\mathbf{tw}(G_S) \leqslant 3$ either i) or ii) holds for G_S.

Suppose that G_S has a tree decomposition (X, T) such that for any bag $X_i \in X$, if $x \in X_i$, then $|X_i| \leqslant 3$. We construct the tree decomposition of G as follows. First we remove r' from all the bags and replace x by s_2, s_3 in every bag. Let i be a node of T with the bag X_i that contain s_2, s_3. We add a path of length three with the nodes i, i_1, i_2, i_3 to T, and set $X_{i_1} = \{s_2, s_3, x_1, x_2\}$, $X_{i_2} = \{s_1, x_1, x_2\}$, and $X_{i_3} = V(F_1)$. We get a tree decomposition of G of width at most three. Notice that if $d_G(r) = 1$, then s_1 is adjacent to r, and s_1 is included in two bags of size at most three.

Suppose now that for any graph G' obtained from $G_S - r'$ by splitting x into z_1, z_2, $\mathbf{tw}(G') \leqslant 3$ and there is a tree decomposition (X, T) of G' of width at most three such that there is a bag $X_i \in X$ with $z_1, z_2 \in X_i$. Then there is a tree decomposition (X, T) of F_2 of width at most three such that there is a bag $X_i \in X$ with $s_2, s_3 \in X_i$. We add a path of length three with the nodes i, i_1, i_2, i_3 to T, and set $X_{i_1} = \{s_2, s_3, x_1, x_2\}$, $X_{i_2} = \{s_1, x_1, x_2\}$, and $X_{i_3} = V(F_1)$. We get a tree decomposition of G of width at most three. Notice that if $d_G(r) = 1$, then s_1 is adjacent to r, and s_1 is included in two bags of size at most three.

Case 3.d. $H = H_4$. Without loss of generality we assume that $s_1, s_3 \in V(F_1)$ and $s_2, s_4 \notin V(F_1)$. Otherwise, either a copy of H_1 in H is a non-separating subgraph of F_1 and $F_1 - H_1$ is a connected odd graph with an even number of edges, or we can find a separating $H = H_1$ with the same component F_1. Denote by x_1, x_2 the non-terminal vertices of H adjacent to s_1, s_2, and let y_1, y_2 be the non-terminal vertices of H adjacent to s_3, s_4.

Suppose that s_2, s_4 belong to different components F_2, F_3 of G'. Then one of these components, say F_3, contains an odd number of edges. Assume that $s_4 \in V(F_3)$. Then the subgraph with the vertex set $V(F_1) \cup V(F_2) \cup \{x_1, x_2\}$ and the edge set $E(F_1) \cup E(F_2) \cup \{s_1x_1, s_1x_2, x_1x_2, x_1s_2, x_2s_2\}$ (see Fig. 24a) is a connected odd graph with an even number of edges, and it contradicts the minimality of G. Hence, s_2, s_4 are vertices of one component of G' with an odd number of edges, and we denote it by F_2 (see Fig. 24b).

We apply Lemma 5 for $W = \{s_1, s_3\}$. Since F_2 is connected, F_1 is a minimal (not necessarily connected) odd graph with an odd number of edges such that $r \in V(G)$, $W \subseteq V(G)$, and each component of G contains at least one vertex of W. By minimality, F_1 has no connected odd subgraph with an even number of edges rooted in r. By the choice of H, F_1 cannot contain a separating $H' \in \{H_1, H_2, H_3\}$ such that for a component F' of $F_1 - H'$ with $r \in V(F')$, $V(F') \cap W = \emptyset$. Suppose that F_1 contains an induced cycle C on three vertices such that the graph obtained from F_1 by the removal of $E(C)$ has two components F_1', F_2' and $V(F_1) \cap W \neq \emptyset$, $V(F_2) \cap W \neq \emptyset$. Let $r, s_1 \in V(F_1')$ and $s_3 \in V(F_2')$. Then the union of C and the triangle $x_1x_2s_2$ (see Fig. 24c) gives a copy of H_6, for which we could get a smaller F_1. It follows that F_1 is a tree, and by minimality and symmetry, F_1 is one of the graphs $F_1^{(1)}, F_1^{(2)}, F_1^{(3)}, F_1^{(4)}$ shown in Fig. 25.

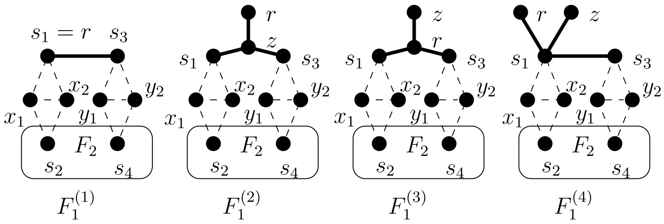

Fig. 25. The trees $F_1^{(1)}, \ldots, F_1^{(4)}$.

We observe that F_2 is a minimal connected odd graph with an odd number of edges containing s_2, s_4. Also F_2 has no connected odd subgraph with an even number of edges rooted in s_2 or in s_4. Indeed, if F_2 contains a connected odd subgraph F with an even number of edges rooted, say, in s_2, then the graph with the vertex set $V(F_1) \cup V(F) \cup \{x_1, x_2\}$ and the edge set $E(F_1) \cup E(F) \cup \{s_1x_1, s_1x_2, x_1x_2, x_1s_2, x_2s_2\}$ is a connected odd subgraph of G with an even number of edges that contains r; a contradiction.

By Lemma 6, $\mathbf{tw}(F_2) \leqslant 2$. Consider a tree decomposition (X, T) of F_2 of width at most two. Let i be a node of T such that $s_2 \in X_i$.

If $F_1 \in \{F_1^{(1)}, F_1^{(2)}, F_1^{(3)}\}$, then we construct a tree decomposition for G from (X, T) as follows. We include vertex s_4 in all the bags.

If $F_1 = F_1^{(1)}$, then we add a path of length five with the nodes $i, i_1, i_2, i_3, i_4, i_5$ to T, and set $X_{i_1} = \{s_2, x_1, x_2, s_4\}$, $X_{i_2} = \{s_1, x_1, x_2, s_4\}$, $X_{i_3} = \{s_1, s_3, s_4\}$, $X_{i_4} = \{s_3, y_1, y_2, s_4\}$, and $X_{i_5} = \{s_4, y_1, y_2\}$.

If $F_1 = F_1^{(2)}$ or $F_1 = F_1^{(3)}$, then we add a path of length six with the nodes $i, i_1, i_2, i_3, i_4, i_5, i_6$ and a node j adjacent to i_4 to T. If $F_1 = F_1^{(2)}$ then $X_{i_1} = \{s_2, x_1, x_2, s_4\}$, $X_{i_2} = \{s_1, x_1, x_2, s_4\}$, $X_{i_3} = \{s_1, z, s_4\}$, $X_{i_4} = \{s_3, z, s_4\}$, $X_{i_5} = \{s_3, y_1, y_2, s_4\}$, $X_{i_6} = \{s_4, y_1, y_2\}$, and $X_j = \{z, r\}$. If $F_1 = F_1^{(3)}$ then $X_{i_1} = \{s_2, x_1, x_2, s_4\}$, $X_{i_2} = \{s_1, x_1, x_2, s_4\}$, $X_{i_3} = \{s_1, r, s_4\}$, $X_{i_4} = \{s_3, r, s_4\}$, $X_{i_5} = \{s_3, y_1, y_2, s_4\}$, $X_{i_6} = \{s_4, y_1, y_2\}$, and $X_j = \{z, r\}$. We get a tree decomposition of G of width at most four. Notice that if $d_G(r) = 1$, then z is adjacent to r, and z is included in three bags of size at most three.

Now, let $F_1 = F_1^{(4)}$. We prove that for any graph G' obtained from $G - r$ by splitting s_1 into w_1, w_2, $\mathbf{tw}(G') \leqslant 3$ and there is a tree decomposition of G' of width at most three such that there is a bag that includes w_1, w_2. Assume that the vertex w_2 is adjacent to s_3 in G'. We consider (X, T) and include the vertex s_4 in all the bags. We add a path of length five with the nodes $i, i_1, i_2, i_3, i_4, i_5$ and then a path i_2, j_1, j_2 to T. Then $X_{i_1} = \{s_2, x_1, x_2, s_4\}$, $X_{i_2} = \{w_2, x_1, x_2, s_4\}$, $X_{i_3} = \{w_2, s_3, s_4\}$, $X_{i_4} = \{s_3, y_1, y_2, s_4\}$, $X_{i_5} = \{s_4, y_1, y_2\}$, and $X_{j_1} = \{w_1, w_2, x_1, x_2\}$, $X_{j_2} = \{w_1, w_2, z\}$.

Case 3.e. $H = H_5$. Denote by x_1, x_2 the non-terminal vertices of H_2 adjacent to s_2, and let y_1, y_2 be the non-terminal vertices of H_3 adjacent to s_3. We assume without loss of generality that either $s_1 \in V(F_1)$ and $s_2, s_3 \notin V(F_1)$ or $s_1 \notin V(F_1)$ and $s_2, s_3 \in V(F_1)$.

Suppose that $s_1 \in V(F_1)$ and $s_2, s_3 \notin V(F_1)$. Then by the same arguments as in Case 3.d, we prove that s_2, s_3 are vertices of one component F_2 and $\mathbf{tw}(F_2) \leqslant 2$. Then exactly as in Case 3.a we prove that $F_1 = K_2$. Let (X, T) be a tree decomposition of F_2 of width at most two, and let i be a node of T such that $s_2 \in X_i$.

If $r = s_1$ then we construct a tree decomposition for G as follows. We consider (X, T) and include the vertex s_3 in all the bags. We add a path of length four with the nodes i, i_1, i_2, i_3, i_4 and a node j adjacent to i_2 to T. Let $X_{i_1} = \{s_2, x_1, x_2, s_3\}$, $X_{i_2} = \{s_1, x_1, x_2, s_3\}$, $X_{i_3} = \{s_1, y_1, y_2, s_3\}$, $X_{i_4} = \{s_3, y_1, y_2\}$, and $X_j = V(F_1)$. We get a tree decomposition of G of width at most three.

Suppose that $r \neq s_1$. We prove that for any graph G' obtained from $G - r$ by splitting s_1 into w_1, w_2, $\mathbf{tw}(G') \leqslant 3$ and there is a tree decomposition of G' of width at most three such that there is a bag that includes w_1, w_2. We consider (X, T) and include the vertex s_4 in all the bags. By the symmetry, it is sufficient to consider three cases.

1. w_1 is adjacent to x_1 and w_2 is adjacent to x_2, y_1, y_2 (see Fig. 26a). We add a path of length four with the nodes i, i_1, i_2, i_3, i_4 and a node j adjacent to i_2 to T. Let $X_{i_1} = \{s_2, x_1, x_2, s_3\}$, $X_{i_2} = \{w_2, x_1, x_2, s_3\}$, $X_{i_3} = \{w_1, y_1, y_2, s_3\}$, $X_{i_4} = \{s_3, y_1, y_2\}$, and $X_j = \{w_1, w_2, x_1, s_3\}$.
2. w_1 is adjacent to x_1, x_2 and w_2 is adjacent to y_1, y_2 (see Fig. 26b). We add a path of length five with the nodes $i, i_1, i_2, i_3, i_4, i_5$ to T. Let $X_{i_1} = \{s_2, x_1, x_2, s_3\}$, $X_{i_2} = \{w_1, x_1, x_2, s_3\}$, $X_{i_3} = \{w_1, w_2, s_3\}$, $X_{i_3} = \{w_2, y_1, y_2, s_3\}$, $X_{i_4} = \{s_3, y_1, y_2\}$.
3. w_1 is adjacent to x_1, y_1 and w_2 is adjacent to x_2, y_2 (see Fig. 26c). We add a path of length six with the nodes $i, i_1, i_2, i_3, i_4, i_5, i_6$ to T. Let $X_{i_1} = \{s_2, x_1, x_2, s_3\}$, $X_{i_2} = \{w_1, x_1, x_2, s_3\}$, $X_{i_3} = \{w_1, w_2, x_2, s_3\}$, $X_{i_4} = \{w_1, w_2, y_1, s_3\}$, $X_{i_5} = \{w_2, y_1, y_2, s_3\}$, $X_{i_6} = \{s_3, y_1, y_2\}$.

Assume now that $s_1 \notin V(F_1)$ and $s_2, s_3 \in V(F_1)$. Then we observe that by minimality $F_2 = K_2$, and by the same arguments as in Case 3d we prove that G is one of the graphs shown in Fig. 26 It is straightforward to see that the claim of the lemma holds in this case.

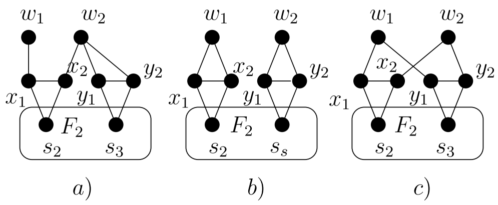

Fig. 26. Splitting of s_1.

Case 3.f. $H = H_6$. By the previous cases, we can assume that G does not have a copy of H_4 or H_5 containing H. Suppose that for one triangle in H, say $s_1s_2s_3$, $s_1, s_2, s_3 \in V(F_1)$. Then the set of edges $\{s_1s_2, s_2s_3, s_1s_3\}$ is not a cut-set in F_1, and the graph obtained from F_1 by the addition of these edges is a connected odd graph with an even number of edges; a contradiction.

Suppose that for one triangle in H, say $s_4s_5s_6$, $s_4, s_5, s_6 \notin V(F_1)$. Then let $C = s_1s_2s_3$ and consider the graph G' obtained from G by the removal of $E(C)$. The graph G' is not connected and F_1 is a component of G'.

Let F_1 include the unique vertex from C and assume that $s_1 \in V(F_1)$. If we can choose distinct vertices $u_1, u_2 \neq s_1$ adjacent to s_2, s_3 respectively, then the subgraph of G with the vertex set $V(F_1) \cup \{s_2, s_3, u_1, u_2\}$ and the edge set $E(F_1) \cup \{s_1s_2, s_2s_3, s_1s_3, s_2u_1, s_3u_2\}$ is a connected odd graph with an even number of edges; a contradiction. Hence, $d_G(s_2) = d_G(s_3) = 3$ and s_2, s_3 have the unique neighbor $u \neq s_1$. But in this case we should choose H_1 instead H_6.

Now, let now F_1 contain two vertices from C and assume that $s_1, s_2 \in V(F_1)$. There is a vertex $u \neq s_1, s_2$ adjacent to s_3. We apply Lemma 5 for $W = \{s_1, s_2\}$. Clearly F_1 is a minimal odd graph with an odd number of edges such that $r \in V(G)$, $W \subseteq V(G)$, and each component of G contains at least one vertex of W. By minimality, F_1 has no connected odd subgraph with an even number of edges rooted in r. By the choice of H, F_1 cannot contain a separating $H' \in \{H_1, H_2, H_3\}$ such that for a component F' of $F_1 - H'$ with $r \in V(F')$, $V(F') \cap W = \emptyset$. Suppose that F_1 contains an induced cycle C' on three vertices such that the graph obtained from F_1 by the removal of $E(C')$ has two components F_1', F_2' and $V(F_1) \cap W \neq \emptyset$, $V(F_2) \cap W \neq \emptyset$. Then the subgraph of G with the vertex set $V(F_1') \cup V(F_2') \cup \{s_3, u\}$ and the edge set $E(F_1') \cup E(F_2') \cup \{s_1s_2, s_2s_3, s_1s_3, s_3u\}$ is a connected odd graph with an even number of edges; a contradiction. Hence, F_1 is a tree, but by minimality, we can find a copy of H_1 that should be chosen instead of H_6.

From now we assume that each triangle in H has at least one vertex in F_1 and at least one vertex not in F_1.

Suppose that F_1 contains a single vertex of each triangle in H. We assume that $s_1, s_4 \in V(F_1)$. The considered graph $H = H_6$ is not a subgraph of H_4 or H_5. Hence, there are distinct vertices $u_1, u_2 \notin V(F_1)$ adjacent to either s_2, s_3 respectively or to s_5, s_6 respectively. Assume that $u_1s_2, u_2s_3 \in E(G)$. Then the subgraph of G with the vertex set $V(F_1) \cup \{s_2, s_3, u_1, u_2\}$ and the edge set $E(F_1) \cup \cup\{s_1s_2, s_2s_3, s_1s_3\} \cup \{s_2u_1, s_3u_2\}$ is an odd graph with en even number of edges; a contradiction.

Suppose that F_1 contains a single vertex from $\{s_1, s_2, s_3\}$ and two vertices from $\{s_4, s_5, s_6\}$. We assume that $s_1, s_4, s_5 \in V(F_1)$.

If we can choose distinct vertices $u_1, u_2 \neq s_1$ adjacent to s_2, s_3 respectively, then the subgraph of G with the vertex set $V(F_1) \cup \{s_2, s_3, u_1, u_2\}$ and the edge set $E(F_1) \cup \{s_1s_2, s_2s_3, s_1s_3, s_2u_1, s_3u_2\}$ is a connected odd graph with an even number of edges; a contradiction. Hence, $d_G(s_2) = d_G(s_3) = 3$ and s_2, s_3 have the unique neighbor $u \neq s_1$.

Let F_2 be the graph obtained from G by the removal of $V(F_1) \cup \{s_2, s_3\}$. Exactly as in Case 3.d we show that F_2 is a connected odd graph with an odd number of edges (see Fig. 27a).

Now we apply Lemma 5 for $W = \{s_1, s_4, s_5\}$. Clearly F_1 is a minimal odd graph with an odd number of edges such that $r \in V(G)$, $W \subseteq V(G)$, and each component of G contains at least one vertex of W. By minimality, F_1 has no connected odd subgraph with an even number of edges rooted in r. By the choice of H, F_1 cannot contain a separating $H' \in \{H_1, H_2, H_3\}$ such that for a component F' of $F_1 - H'$ with $r \in V(F')$, $V(F') \cap W = \emptyset$.

Suppose that F_1 contains an induced cycle $C = z_1z_2z_3$ on three vertices such that the graph obtained from F_1 by the removal of $E(C)$ has two components F_1', F_2' and $V(F_1) \cap W \neq \emptyset$, $V(F_2) \cap W \neq \emptyset$. Let $r \in V(F_1')$. If $s_4 \in V(F_1'), s_5 \in V(F_2')$ or $s_5 \in V(F_1'), s_4 \in V(F_2')$, then the subgraph obtained from G by the removal of s_2, s_3 and $E(C)$ a connected odd graph with an even number of edges (see Fig. 27b); a contradiction. Therefore, either $s_1 \in V(F_1')$ and $s_4, s_5 \in V(F_2')$ or $s_4, s_5 \in V(F_1')$ and $s_1 \in V(F_2')$. If $s_1 \in V(F_1')$ and $s_4, s_5 \in V(F_2')$, then we observe that the union of C and the triangle s_2s_3u could be chosen instead of H (see Fig. 27c). Hence, $s_4, s_5 \in V(F_1')$ and $s_1 \in V(F_2')$. If the triangles C and $s_4s_5s_6$ are disjoint, then their union could be chosen instead of H. It means that these triangles have a common vertex. Assume that $z_1 = s_5$ and $z_2 \in V(F_2')$. Observe that F_1' is an odd graph with an odd number of edges. Then the subgraph of G with the vertex set $V(F_1') \cup \{s_6, z_2\}$ and the edge set $E(F_1') \cup \{s_6s_4, s_4s_5, s_5z_2\}$ (see Fig. 27d) is a connected odd graph with an even number of edges; a contradiction.

We conclude that F_1 is a tree. By minimality, $d_G(s_4) = d_G(s_5) = 3$ and F_1 has the unique vertex w adjacent to s_4, s_5. But then we have the two copies of H_1 induced by the sets s_1, s_2, s_3, u and s_4, s_5, s_6, u respectively. Hence, we should choose either $H = H_4$ or $H = H_5$ instead of H_6.

It remains to consider the final case when F_1 contains two vertices from $\{s_1, s_2, s_3\}$ and two vertices from $\{s_4, s_5, s_6\}$. We assume that $s_1, s_2, s_4, s_5 \in V(F_1)$.

Let F_2 be the graph obtained from G by the removal of $V(F_1)$. Exactly as in Case 3d we show that F_2 is a connected odd graph with an odd number of edges.

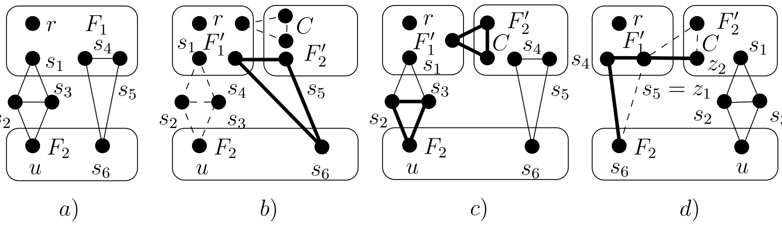

Fig. 27. The case $s_1, s_4, s_5 \in V(F_1)$.

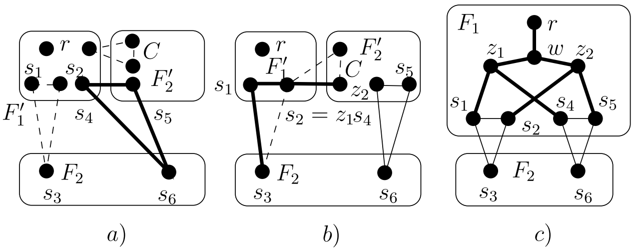

Fig. 28. The case $s_1, s_2, s_4, s_5 \in V(F_1)$.

Now we apply Lemma 5 for $W = \{s_1, s_2, s_4, s_5\}$. Clearly F_1 is a minimal odd graph with an odd number of edges such that $r \in V(G)$, $W \subseteq V(G)$, and each component of G contains at least one vertex of W. By minimality, F_1 has no connected odd subgraph with an even number of edges rooted in r. By the choice of H, F_1 cannot contain a separating $H' \in \{H_1, H_2, H_3\}$ such that for a component F' of $F_1 - H'$ with $r \in V(F')$, $V(F') \cap W = \emptyset$.

Suppose that F_1 contains an induced cycle $C = z_1z_2z_3$ on three vertices such that the graph obtained from F_1 by the removal of $E(C)$ has two components F_1', F_2' and $V(F_1) \cap W \neq \emptyset$, $V(F_2) \cap W \neq \emptyset$. Let $r \in V(F_1')$. If $s_4 \in V(F_1'), s_5 \in V(F_2')$ or $s_5 \in V(F_1'), s_4 \in V(F_2')$, then the subgraph obtained from G by the removal of $E(C) \cup \{s_1s_2, s_2s_3, s_1s_3\}$ (see Fig. 28a) is a connected odd graph with an even number of edges; a contradiction. Similarly, if $s_1 \in V(F_1'), s_2 \in V(F_2')$ or $s_2 \in V(F_1'), 1_4 \in V(F_2')$, then the subgraph obtained from G by the removal of $E(C) \cup \{s_4s_5, s_5s_6, s_4s_6\}$ a connected odd graph with an even number of edges; a contradiction. Therefore, we can assume that $s_1, s_2 \in V(F_1')$ and $s_4, s_5 \in V(F_2')$. If the triangles C and $s_1s_2s_3$ are disjoint, then their union could be chosen instead of H. It means that these triangles have a common vertex. Assume that $z_1 = s_2$ and $z_2 \in V(F_2')$. Observe that F_1' is an odd graph with an odd number of edges. Then the subgraph of G with the vertex set $V(F_1') \cup \{s_3, z_2\}$ and the edge set $E(F_1') \cup \{s_3s_1, s_1s_2, s_2z_2\}$ (see Fig. 28b) is a connected odd graph with an even number of edges; a contradiction.

We conclude that F_1 is a tree. By minimality and the choice of H, it should be a tree shown in Fig. 28c.

Observe that F_2 is a minimal connected odd graph with an odd number of edges that contains s_3, s_6. Also F_2 has no connected odd subgraph with an even number of edges rooted in s_3 or in s_6. Indeed, if F_2 had a connected odd subgraph F with an even number of edges rooted, say, in s_3, then the graph with the vertex set $V(F_1) \cup V(F)$ and the edge set $E(F_1) \cup E(F) \cup \{s_1s_2, s_2s_3, s_1s_3\}$ is a connected odd subgraph of G with an even number of edges that contains r; a contradiction.

By Lemma 6, $\mathbf{tw}(F_2) \leqslant 2$. Consider a tree decomposition of F_2 of width at most two. Let i a node of T such that $s_3 \in X_i$. We construct a tree decomposition for G as follows. We include the vertex s_6 in all the bags. We add a path of length six with the nodes $i, i_1, i_2, i_3, i_4, i_5, i_6$ and a path $i_3j_1j_2$ to T, and set $X_{i_1} = \{s_1, s_2, s_3, s_6\}$, $X_{i_2} = \{s_1, s_2, z_1, s_6\}$, $X_{i_3} = \{s_2, z_1, z_2, s_6\}$ $X_{i_4} = \{z_1, z_2, s_4, s_6\}$ $X_{i_5} = \{z_2, s_4, s_5, s_6\}$, $X_{i_5} = \{s_4, s_5, s_6\}$, and $X_{j_1} = \{z_1, z_2, w\}$, $X_{j_2} = \{w, r\}$. We get a tree decomposition of G of width at most four. Notice that $d_G(r) = 1$ and w is included in two bags of size at most three.

This completes the case analysis and the proof of Claim 1.

5. Complexity of *k*-Vertex Eulerian Subgraph

In this section we prove that k-Vertex Eulerian Subgraph is W[1]-hard.

Theorem 3. *The k-*Vertex Eulerian Subgraph *is* W[1]*-hard.*

Proof. We reduce from the well-known W[1]-complete k-Clique problem (see e.g. [6]):

k-Clique	
Instance:	A graph G and non-negative integer k.
Parameter:	k.
Question:	Does G contain a clique with k vertices?

Notice that the problem remains W[1]-complete when the parameter k is restricted to be odd. It follows immediately from the observation that the existence of a clique with k vertices in a graph G is equivalent to the existence of a clique with $k+1$ vertices in the graph obtained from G by the addition of a universal vertex adjacent to all the vertices of G. From now it is assumed that $k > 1$ is an odd integer.

Let G be a graph. We construct the graph G' by subdividing edges of G by k^2 vertices, i.e. each edge xy is replaced by an (x, y)-path of length $k^2 + 1$. We say that $u \in V(G')$ is a *branch* vertex if $u \in V(G)$, and u is a *subdivision* vertex otherwise. We also say that u is a *subdivision vertex for an edge* $xy \in E(G)$ if u is a subdivision vertex of the path obtained from xy. We claim that G has a clique of size k if and only if G' has an induced Eulerian subgraph on $k' = \frac{1}{2}(k-1)k^3 + k$ vertices.

Suppose that G has a clique K with k vertices. Let H be the subgraph of G induced by K and the subdivision vertices for all edges xy with $x, y \in K$. It is easy to see that H is a connected Eulerian graph on $k' = \frac{1}{2}(k-1)k^3 + k$ vertices.

Now, let H be an induced Eulerian subgraph of G' on $k' = \frac{1}{2}(k-1)k^3 + k$ vertices. Denote by U the set of branch vertices of H, and let $p = |U|$. Let $A = \{xy \in E(G) | x, y \in U,$ and H has a subdivision vertex for $xy\}$ and let $F = (U, A)$. Also, let q denote $|A|$. Since H is connected, the graph F is connected as well. Observe that if $u \in V(H)$ is a subdivision vertex for an edge $xy \in E(G)$, then all subdivision vertices for xy are vertices of H and $x, y \in V(H)$. It follows that H has $p + q \cdot k^2 = k'$ vertices, and we have $p - k = (\frac{1}{2}(k-1)k - q)k^2$. Since k^2 is a divisor of $p - k$, $p \geqslant k$. Suppose that $p > k$. Then since k^2 is a divisor of $p - k$, $p \geqslant k^2 + k$. Any connected graph with p vertices has at least $p - 1$ edges, and it means that $q \geqslant k^2 + k - 1 > \frac{1}{2}(k-1)k$. We get that $0 < p - k = (\frac{1}{2}(k-1)k - q)k^2 < 0$; a contradiction. We conclude that $p = k$. Then $q = \frac{1}{2}(k-1)k$ and U is a clique with k vertices. $\square$

Recall that k-Vertex Eulerian Subgraph asks about an *induced* Eulerian subgraph on k vertices. For the graph G' in the proof of Theorem 3, any Eulerian subgraph is induced. It gives us the following corollary.

Corollary 2. *The following problem*:

Instance:	*A graph G and non-negative integer k.*
Parameter:	*k.*
Question:	*Does G contain an Eulerian* (*not necessarily induced*) *subgraph with k vertices*?

is W[1]*-hard.*

6. Conclusion

We proved that k-Edge Connected Odd Subgraph is FPT and k-Vertex Eulerian Subgraph is W[1]-hard. This completes the characterization of even/odd subgraph problems with *exactly* k edges or vertices from parameterized complexity perspective. While it is trivial to decide whether a graph G has a (connected) even or odd subgraph with *at most* k edges or vertices, the question about a subgraph with *at least* k edges or vertices seems to be much more complicated. For At Least k-Edge Odd Subgraph and At Least k-Vertex Odd Subgraph, following the lines of the proofs from [4] for k-Edge Odd Subgraph and k-Vertex Odd Subgraph, it is possible to show that these problems are in FPT. For other cases, the approaches used in [4] and in our paper, do not seem to work.

Cai and Yang in [4] also considered dual problems where the aim is to find an even or odd subgraph of a graph G with $|V(G)| - k$ vertices or $|E(G)| - k$ edges respectively. Recently, these results were complemented by Cygan et al. [5]. However, the complexity of the dual problem to k-Edge Connected Odd Subgraph, namely, obtaining connected odd subgraph with $|E(G)| - k$ edges, remains open.

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