

A SUBEXPONENTIAL PARAMETERIZED ALGORITHM FOR PROPER INTERVAL COMPLETION*

IVAN BLIZNETS[†], FEDOR V. FOMIN[‡], MARCIN PILIPCZUK[§], AND
MICHAL PILIPCZUK[§]

Abstract. In the PROPER INTERVAL COMPLETION problem we are given a graph G and an integer k , and the task is to turn G using at most k edge additions into a proper interval graph, i.e., a graph admitting an intersection model of equal-length intervals on a line. The study of PROPER INTERVAL COMPLETION from the viewpoint of parameterized complexity has been initiated by Kaplan, Shamir, and Tarjan [*SIAM J. Comput.*, 28 (1999), pp. 1906–1922], who showed an algorithm for the problem working in $\mathcal{O}(16^k \cdot (n + m))$ time. In this paper we present an algorithm with running time $k^{\mathcal{O}(k^{2/3})} + \mathcal{O}(nm(kn + m))$, which is the first subexponential parameterized algorithm for PROPER INTERVAL COMPLETION.

Key words. fixed-parameter tractability, proper interval graphs, proper interval completion, subexponential algorithm

AMS subject classifications. 68Q25, 68R10, 68W40, 05C85

DOI. 10.1137/140988565

1. Introduction. A graph G is an *interval graph* if it admits a model of the following form: each vertex is associated with an interval on the real line, and two vertices are adjacent if and only if the associated intervals overlap. If moreover the intervals can be assumed to be of equal length, then G is a *proper interval graph*; equivalently, one may require that no associated interval is contained in another [18]. Interval and proper interval graphs appear naturally in molecular biology in the problem of *physical mapping*, where one is given a graph with vertices modeling contiguous intervals (called *clones*) in a DNA sequence, and the edges indicate which intervals overlap. Based on this information one would like to reconstruct the layout of the clones. We refer to [11, 12, 14] for further discussion on biological applications of (proper) interval graphs.

The biological motivation was the starting point of the work of Kaplan, Shamir, and Tarjan [14], who initiated the study of (proper) interval graphs from the point of view of parameterized complexity. It is natural to expect that some information about overlaps will be lost, and hence the model will be missing a small number of edges. Thus we arrive at the problems of INTERVAL COMPLETION and PROPER INTERVAL COMPLETION: given a graph G and an integer k , one is asked to add at

*Received by the editors September 23, 2014; accepted for publication (in revised form) August 18, 2015; published electronically October 22, 2015. The research leading to these results received funding from the European Research Council under the European Union’s Seventh Framework Programme (FP/2007-2013)/ERC grant agreement 267959 as well as from the government of the Russian Federation (grant 14.Z50.31.0030) and a grant from the president of the Russian Federation (MK-6550.2015.1). This research was done while the third and fourth authors were at the Department of Informatics, University of Bergen, Norway. A preliminary version of this work was presented at ESA 2014.

<http://www.siam.org/journals/sidma/29-4/98856.html>

[†]Steklov Institute of Mathematics at St. Petersburg, Russian Academy of Sciences, St. Petersburg, Russia (iabliznets@gmail.com).

[‡]Department of Informatics, University of Bergen, Bergen, Norway (fomin@ii.uib.no).

[§]Institute of Informatics, University of Warsaw, Warsaw, Poland (marcin.pilipczuk@mimuw.edu.pl, michal.pilipczuk@mimuw.edu.pl).

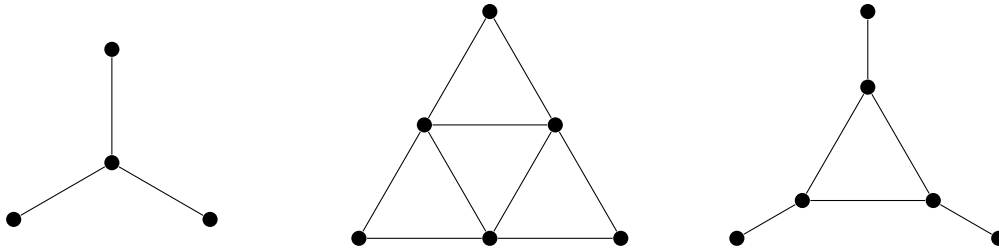


FIG. 1. *Small forbidden induced subgraphs for proper interval graphs: a claw, a tent, and a net.*

most k edges to G to obtain a (proper) interval graph. Both problems are known to be NP-hard [20], and hence it is natural to ask for a fixed-parameter tractable (FPT) algorithm parameterized by the expected number of additions k . For PROPER INTERVAL COMPLETION Kaplan, Shamir, and Tarjan [14] presented an algorithm with running time $\mathcal{O}(16^k \cdot (n + m))$, while fixed-parameterized tractability of INTERVAL COMPLETION was resolved much later by Villanger et al. [19]. Recently, Liu et al. [16] obtained an $\mathcal{O}(4^k + nm(n + m))$ -time algorithm for PIC.

The approach of Kaplan, Shamir, and Tarjan [14] is based on a characterization by *forbidden induced subgraphs*, also studied by Cai [5]: proper interval graphs are exactly graphs that are chordal, i.e., do not contain any induced cycle C_ℓ for $\ell \geq 4$, and moreover exclude three special structures as induced subgraphs: a *claw*, a *tent*, and a *net* (see Figure 1). Therefore, when given a graph which is to be completed into a proper interval graph, we may apply a basic branching strategy. Whenever a forbidden induced subgraph is encountered, we branch into several possibilities of how it is going to be destroyed in the optimal solution. A cycle C_ℓ can be destroyed only by triangulating it, which requires adding exactly $\ell - 3$ edges and can be done in roughly $4^{\ell-3}$ different ways. Since for special structures there is only a constant number of ways to destroy them, the whole branching procedure runs in $c^k n^{\mathcal{O}(1)}$ time for some constant c .

The approach via forbidden induced subgraphs has driven the research on the parameterized complexity of graph modification problems ever since the work of Cai [5]. Of particular importance was the work on polynomial kernelization; recall that a *polynomial kernel* for a parameterized problem is a polynomial-time preprocessing routine that shrinks the size of the instance at hand to polynomial in the parameter. While many natural completion problems admit polynomial kernels, there are also examples where no polynomial kernel exists under plausible complexity assumptions [15]. In particular, PROPER INTERVAL COMPLETION admits a kernel with $\mathcal{O}(k^3)$ vertices which can be computed in $\mathcal{O}(nm(kn + m))$ time [2], while the kernelization status of INTERVAL COMPLETION remains a notorious open problem.

The turning point came recently, when Fomin and Villanger [9] proposed an algorithm for FILL-IN, i.e., CHORDAL COMPLETION, that runs in *subexponential parameterized time*, more precisely, $k^{\mathcal{O}(\sqrt{k})} n^{\mathcal{O}(1)}$. As observed in [14], the approach via forbidden induced subgraphs leads to an FPT algorithm for FILL-IN with running time $16^k n^{\mathcal{O}(1)}$. Observe that in order to achieve a subexponential running time one needs to completely abandon this route, as even branching on encountered obstacles as small as, say, induced C_4 -s leads to running time at least $2^k n^{\mathcal{O}(1)}$. To circumvent this, Fomin and Villanger proposed the approach of gradually building the struc-

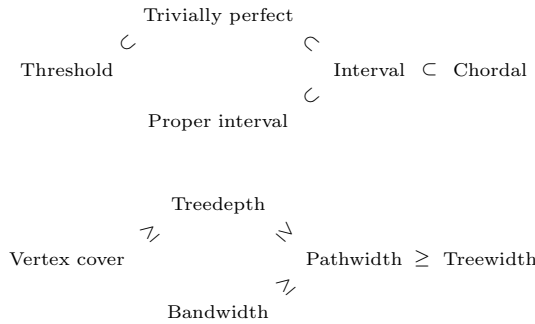


FIG. 2. Graph classes and corresponding graph parameters. Inequalities on the bottom diagram are with ± 1 slackness.

ture of a chordal graph in a dynamic programming manner. The crucial observation was that the number of “building blocks” (in their case, *potential maximal cliques*) is subexponential in a YES-instance, and thus the dynamic program operates on a subexponential space of states.

This research direction was continued by Ghosh et al. [10] and by Drange et al. [7], who identified several more graph classes for which completion problems have subexponential parameterized complexity: threshold graphs, split graphs, pseudo-split graphs, and trivially perfect graphs (we refer to [7, 10] for respective definitions). Let us remark that problems admitting subexponential parameterized algorithms are very scarce, since for most natural parameterized problems existence of such algorithms can be refuted under the exponential time hypothesis (ETH) [13]. Until very recently, the only natural positive examples were problems on specifically constrained inputs, like H -minor free graphs [6] or tournaments [1]. Thus, completion problems admitting subexponential parameterized algorithms can be regarded as “singular points on the complexity landscape.” Indeed, Drange et al. [7] complemented their work with a number of lower bounds excluding (under ETH) subexponential parameterized algorithms for completion problems to related graphs classes, for instance, cographs.

Interestingly, threshold graphs, trivially perfect graphs, and chordal graphs, which are currently our main examples, correspond to graph parameters *vertex cover*, *treedepth*, and *treewidth* in the following sense: the parameter is equal to the minimum possible maximum clique size in a completion to the graph class (± 1); see Figure 2. It is therefore natural to ask if INTERVAL COMPLETION and PROPER INTERVAL COMPLETION, which likewise correspond to *pathwidth* and *bandwidth*, also admit subexponential parameterized algorithms.

Our results. In this paper we answer the question about PROPER INTERVAL COMPLETION in the affirmative by proving the following theorem.

THEOREM 1.1. PROPER INTERVAL COMPLETION can be solved in $k^{\mathcal{O}(k^{2/3})} + \mathcal{O}(nm(kn + m))$ time.

In case of a positive answer, our algorithm can provide a feasible solution (a set of edges to add to the graph) in the same asymptotic running time.

In a companion paper [3] we also present an algorithm for INTERVAL COMPLETION with running time $k^{\mathcal{O}(\sqrt{k})}n^{\mathcal{O}(1)}$, which means that the completion problems for all the classes depicted in Figure 2 in fact do admit subexponential parameterized algorithms. We now describe briefly our techniques employed to prove Theorem 1.1 and main

differences with the work on interval graphs [3].

From a space-level perspective, the approach of both this paper and [3] follows the route laid out by Fomin and Villanger in [9]. That is, we enumerate a subexponential family of potentially interesting building blocks and then try to arrange them into a (proper) interval model with a small number of missing edges using dynamic programming. In both cases, a natural candidate for this building block is the concept of a *cut*: given an interval model of a graph, imagine a vertical line placed at some position x that pins down intervals containing x . A *potential cut* is then a subset of vertices that becomes a cut in some minimal completion to a (proper) interval graph of cost at most k . The starting point of both this work and [3] is enumeration of potential cuts. Using different structural insights into the classes of interval and proper interval graphs, one can show that in both cases the number of potential cuts is at most $n^{\mathcal{O}(\sqrt{k})}$, and they can be enumerated efficiently. Since in the case of proper interval graphs we can start with a cubic kernel given by Bessy and Perez [2], this immediately gives $k^{\mathcal{O}(\sqrt{k})}$ potential cuts for the PROPER INTERVAL COMPLETION problem. In the interval case the question of existence of a polynomial kernel is wide open, and the need for circumventing this obstacle causes severe complications in [3].

Afterward the approaches diverge completely, as it turns out that in both cases the potential cuts are insufficient building blocks to perform dynamic programming, although for very different reasons. For INTERVAL COMPLETION the problem is that the cut itself does not define what lies on the left and on the right of it. Even worse, there can be an exponential number of possible left/right alignments when the graph contains many modules that neighbor the same clique. To cope with this problem, the approach taken in [3] remodels the dynamic programming routine so that, in some sense, the choice of left/right alignment is taken care of inside the dynamic program. The dynamic programming routine becomes thus much more complicated, and a lot of work needs to be put into bounding the number of its states, which can be very roughly viewed as quadruples of cuts enriched with an “atomic” left/right choice (see the definition of a *nested terrace* in [3]).

Curiously, in the proper interval setting the left/right choice can be easily guessed along with a potential cut at basically no extra cost. Hence, the issue causing the most severe problems in the interval case is simply nonexistent. The problem, however, is in the *ordering* of intervals in the cut: while performing a natural left-to-right dynamic program that builds the model, we would need to ensure that intervals participating in a cut begin in the same order as they end. Therefore, apart from the cut itself and a partition of the other vertices into left and right, we would need to include in a state also the ordering of the vertices of the cut; as the cut may be very large, we cannot afford constructing a state for every possible ordering.

Instead we remodel the dynamic program, this time by introducing two layers. We first observe that the troublesome ordering may be guessed expeditiously providing that the cut in question has only a sublinear in k number of incident edge additions. Hence, in the first layer of dynamic programming we aim at chopping the optimally completed model using such cheap cuts, and to conclude the algorithm we just need to be able to compute the best possible completed model between two border cuts that are cheap, assuming that all the intermediate cuts are expensive. This task is performed by the layer-two dynamic program. The main observation is that since all the intermediate cuts are expensive, there cannot be many disjoint such cuts and consequently the space between the border cuts is in some sense “short.” As the border cuts can be large, it is natural to start partitioning the space in between “horizontally”

instead of “vertically” — shortness of this space guarantees that the number of sensible “horizontal” separations is subexponential. The horizontal partitioning method that we employ resembles the classic $\mathcal{O}^*(10^n)$ exact algorithm for bandwidth of Feige [8].

2. Preliminaries.

Graph notation. In most cases, we follow standard graph notation.

An *ordering* of a vertex set of a graph G is a bijection $\sigma : V(G) \rightarrow \{1, 2, \dots, |V(G)|\}$. We say that a vertex v is *to the left of* or *before* a vertex w if $\sigma(v) < \sigma(w)$ and *to the right of* or *after* w if $\sigma(v) > \sigma(w)$. We also extend these notions to orderings of subsets of vertices: for any $X \subseteq V(G)$, any injective function $\sigma : X \rightarrow \{1, 2, \dots, |V(G)|\}$ is called an ordering. We sometimes treat such σ as an ordering of the vertex set of $G[X]$ as well, implicitly identifying $\sigma(X)$ with $\{1, 2, \dots, |X|\}$ in the monotonous way.

For any graph G we shall speak about, we implicitly fix one arbitrary ordering σ_0 on $V(G)$. We shall use this ordering to break ties and canonize some objects (orderings, completion sets, solutions, etc.). That is, assume that $X = \{x_1, x_2, \dots, x_{|X|}\} \subseteq V(G)$ with $\sigma_0(x_1) < \sigma_0(x_2) < \dots < \sigma_0(x_{|X|})$. Then with every ordering $\sigma : X \rightarrow \{1, 2, \dots, |V(G)|\}$ we associate a sequence $(\sigma(x_1), \sigma(x_2), \dots, \sigma(x_{|X|}))$ and sort the orderings of X according to this sequence lexicographically. In many places we consider some family of orderings for a fixed choice of X ; if we pick the lexicographically minimum ordering of this family, we mean the one with lexicographically minimum associated sequence.

Observe that an ordering σ of $V(G)$ naturally defines a graph $\sigma(G)$ with vertex set $\{1, 2, \dots, |V(G)|\}$ and $pq \in E(\sigma(G))$ if and only if $\sigma^{-1}(p)\sigma^{-1}(q) \in E(G)$. Clearly, $\sigma(G)$ and G are isomorphic with σ being an isomorphism between them.

For any integers a, b we denote $[a, b] = \{a, a + 1, \dots, b\}$.

We use n and m to denote the number of vertices and edges of the input graph.

Proper interval graphs. A graph G is a *proper interval graph* if it admits an intersection model, where each vertex is assigned a closed interval on a line such that no interval is a proper subset of another one, and two vertices are adjacent if and only if their intervals intersect. In our work it is more convenient to use an equivalent combinatorial object, called an *umbrella ordering*.

DEFINITION 2.1 (umbrella ordering). *Let G be a graph and $\sigma : V(G) \rightarrow \{1, 2, \dots, n\}$ be an ordering of its vertices. We say that σ satisfies the umbrella property for a triple $a, b, c \in V(G)$ if $ac \in E(G)$ and $\sigma(a) < \sigma(b) < \sigma(c)$ implies $ab, bc \in E(G)$. Furthermore, σ is called an umbrella ordering if it satisfies the umbrella property for any $a, b, c \in V(G)$. The following result is due to Looges and Olariu.*

THEOREM 2.2 (see [17]). *A graph is a proper interval graph if and only if it admits an umbrella ordering.*

Observe that we may equivalently define an umbrella ordering σ as an ordering such that for every $ab \in E(G)$ with $\sigma(a) < \sigma(b)$ the subgraph $\sigma(G)[[a, b]]$ is a complete graph. Alternatively, σ is an umbrella ordering of G if and only if for any $a, a', b', b \in V(G)$ such that $\sigma(a) \leq \sigma(a') < \sigma(b') \leq \sigma(b)$ and $ab \in E(G)$, it also holds that $a'b' \in E(G)$. We will use these alternative definitions implicitly in what follows. See also Figure 3 for an illustration.

Observe also the following simple fact that follows immediately from the definition of an umbrella ordering.

LEMMA 2.3. *Let G_1, G_2 be two proper interval graphs with $V(G_1) = V(G_2) = V$. Assume further that some ordering σ of V is an umbrella ordering of both G_1 and G_2 . Then σ is also an umbrella ordering of $H_\downarrow := (V, E(G_1) \cap E(G_2))$ and $H_\uparrow := (V, E(G_1) \cup E(G_2))$, and in particular H_\downarrow and H_\uparrow are proper interval graphs.*

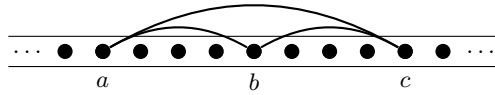


FIG. 3. An umbrella property for triple a, b, c . The existence of an edge ac implies the existence of edges ab and bc .

We use the assumed fixed ordering σ_0 to canonize umbrella orderings: for a proper interval graph G , the canonical umbrella ordering of G is the one with its associated sequence being lexicographically minimum.

Proper interval completion. For a graph G , a *completion* of G is a set $F \subseteq \binom{V(G)}{2} \setminus E(G)$ such that $G + F := (V(G), E(G) \cup F)$ is a proper interval graph. The PROPER INTERVAL COMPLETION problem asks for a completion of G of size not exceeding a given budget k .

However, in our paper it is more convenient to work with orderings as a basic notion, instead of completions. Moreover, for technical reasons, we also need a slightly more general *sandwich* version of the PROPER INTERVAL COMPLETION problem, henceforth called SANDWICH PROPER INTERVAL COMPLETION (SPIC). Here, apart from a graph G and budget k , we are given

1. for each $u \in V(G)$ a set of *allowed positions* $\Sigma_u \subseteq \{1, 2, \dots, |V(G)|\}$;
2. two graphs G_\downarrow and G_\uparrow with vertex set $\{1, 2, \dots, |V(G)|\}$ satisfying
 - (a) G_\downarrow is a subgraph of G_\uparrow ;
 - (b) both G_\downarrow and G_\uparrow are proper interval graphs, and the identity is an umbrella ordering for both of them.

The SPIC problem asks for a completion F of G , together with an ordering σ of $V(G)$, such that

1. σ is an umbrella ordering of $G + F$;
2. $\sigma(u) \in \Sigma_u$ for each $u \in V(G)$;
3. $E(G_\downarrow) \subseteq E(\sigma(G + F)) \subseteq E(G_\uparrow)$;
4. the *cost* of the ordering σ and completion F , defined as $c(\sigma, F) = |F|$, is at most k .

We now observe that an ordering σ in fact yields a unique “best” completion F . Formally, for any ordering σ of $V(G)$ we define F^σ to be the set of such unordered pairs $xy \notin E(G)$ for which one of the following holds:

1. $\sigma(x)\sigma(y) \in E(G_\downarrow)$ or
2. there exist $x', y' \in V(G)$ such that $x'y' \in E(G)$ and $\sigma(x') \leq \min(\sigma(x), \sigma(y)) \leq \max(\sigma(x), \sigma(y)) \leq \sigma(y')$.

We need the following property of F^σ .

LEMMA 2.4. *Set F^σ as a completion of G , σ an umbrella ordering of $G + F^\sigma$, and G_\downarrow a subgraph of $\sigma(G + F^\sigma)$. Furthermore, F^σ is the unique inclusionwise minimal completion of G for which σ is an umbrella ordering of $G + F^\sigma$ and G_\downarrow is a subgraph of $\sigma(G + F^\sigma)$.*

Proof. The claim that G_\downarrow is a subgraph of $\sigma(G + F^\sigma)$ is straightforward from the definition, as we explicitly add the edges of $E(G_\downarrow)$. We now show that σ is an umbrella ordering of $G + F^\sigma$. To this end, consider a triple $a, b, c \in V(G)$ with $\sigma(a) < \sigma(b) < \sigma(c)$ and $ac \in E(G + F^\sigma)$. We consider three cases, depending on the reason why $ac \in E(G + F^\sigma)$.

If $ac \in E(G)$, then, by the second criterion of belonging to F^σ , we have that $ab \in F^\sigma$ unless $ab \in E(G)$ and $bc \in F^\sigma$ unless $bc \in E(G)$. Similarly, if $ac \in F^\sigma$

because of the second criterion for belonging to F^σ , then there exist $a', c' \in V(G)$ with $a'c' \in E(G)$ and $\sigma(a') \leq \sigma(a) < \sigma(c) \leq \sigma(c')$; clearly a', c' also witness that $ab, bc \in E(G) \cup F^\sigma$. Finally, if $\sigma(a)\sigma(c) \in E(G_\downarrow)$, then the assumption that G_\downarrow is a proper interval graph with identity being an umbrella ordering implies that $\sigma(a)\sigma(b) \in E(G_\downarrow)$ and $\sigma(b)\sigma(c) \in E(G_\downarrow)$. Consequently, the umbrella property is satisfied for the triple a, b, c , and σ is an umbrella ordering for $G + F^\sigma$.

To show the second claim of the lemma, simply observe that every completion F of G for which G_\downarrow is a subgraph of $\sigma(G + F^\sigma)$ contains the edges of F^σ falling into the first criterion, whereas every completion F of G for which σ is an umbrella ordering contains the edges of F^σ that fall into the second criterion. \square

Hence, Lemma 2.4 allows us to use the notion of the cost of an ordering σ (instead of the cost of a pair (σ, F) or completion F), where we use the completion F^σ . That is, we denote $c(\sigma) = |F^\sigma|$.

We say that an ordering σ is *feasible* if $\sigma(u) \in \Sigma_u$ for each $u \in V(G)$ and additionally $E(\sigma(G)) \subseteq E(G_\uparrow)$. It is straightforward to verify using Lemma 2.3, minimality of F^σ , and the fact that σ is an umbrella ordering of G_\uparrow that the second condition for σ being feasible is equivalent to $E(\sigma(G + F^\sigma)) \subseteq E(G_\uparrow)$. Hence, by Lemma 2.4, the SPIC problem may equivalently ask for a feasible ordering σ of cost at most k .

Finally, observe that SPIC is a generalization of PROPER INTERVAL COMPLETION, as we may take $\Sigma_u = \{1, 2, \dots, |V(G)|\}$ for each $u \in V(G)$, G_\downarrow to be edgeless and G_\uparrow to be a complete graph. Note that for such an instance, any ordering of $V(G)$ is feasible. In this way, given a PROPER INTERVAL COMPLETION instance (G, k) and an ordering σ of $V(G)$, the notions of F^σ and $c(\sigma)$ are well-defined. Hence, the PROPER INTERVAL COMPLETION problem equivalently asks for an ordering σ of cost at most k , that is, for which $|F^\sigma| \leq k$.

We now set up a few more notions. For a completion F of G and a vertex $v \in V(G)$ by $F(v)$ we denote the set of edges $e \in F$ that are incident with v . We extend this notion to vertex sets $X \subseteq V(G)$ by $F(X) = \bigcup_{v \in X} F(v)$.

For a SPIC instance $(G, k, (\Sigma_u)_{u \in V(G)}, G_\downarrow, G_\uparrow)$ and a feasible ordering σ we denote $G^\sigma := G + F^\sigma$. We extend the notion of feasibility and of F^σ to orderings σ of subsets of $V(G)$ in the following natural manner. If $X \subseteq V(G)$ and $\sigma : X \rightarrow \{1, 2, \dots, |V(G)|\}$ is injective, then σ is feasible if and only if $\sigma(u) \in \Sigma_u$ for each $u \in X$ and $E(\sigma(G[X])) \subseteq E(G_\uparrow)$. The set F^σ is defined as follows: $xy \in F^\sigma$ if and only if $x, y \in X$, $xy \notin E(G)$, but either $\sigma(x)\sigma(y) \in E(G_\downarrow)$ or there exists an edge $x'y' \in E(G[X])$ with $\sigma(x') \leq \min(\sigma(x), \sigma(y)) < \max(\sigma(x), \sigma(y)) \leq \sigma(y')$. Again, the same argument shows that the second condition of feasibility is equivalent to $E(\sigma(G[X] + F^\sigma)) \subseteq E(G_\uparrow)$.

We use the assumed fixed ordering σ_0 to canonize a solution of a SPIC instance $(G, k, (\Sigma_u)_{u \in V(G)}, G_\downarrow, G_\uparrow)$. An ordering σ of $V(G)$ is called the *canonical umbrella ordering* of $(G, k, (\Sigma_u)_{u \in V(G)}, G_\downarrow, G_\uparrow)$ if σ is feasible, its cost is minimum possible, and σ is lexicographically smallest with this property. This notion projects to the notion of a canonical umbrella ordering of a graph G by taking again $\Sigma_u = \{1, 2, \dots, n\}$ for any $u \in V(G)$, G_\downarrow to be edgeless, and G_\uparrow to be a complete graph. Observe that this notion thus extends the notion of canonical umbrella ordering for proper interval graphs, and as in the case of a proper interval graph the unique minimum completion is empty.

The associated completion F^σ with the canonical umbrella ordering σ is called the *canonical completion*. If additionally the cost of σ is at most k , we call σ the

canonical solution to the SPIC instance $(G, k, (\Sigma_u)_{u \in V(G)}, G_\downarrow, G_\uparrow)$ or, in the special case, to a PROPER INTERVAL COMPLETION instance (G, k) .

A polynomial kernel. Our starting point for the proof of Theorem 1.1 is the polynomial kernel for PROPER INTERVAL COMPLETION due to Bessy and Perez.

THEOREM 2.5 (see [2]). PROPER INTERVAL COMPLETION admits a kernel with $\mathcal{O}(k^3)$ vertices computable in time $\mathcal{O}(nm(kn + m))$.

That is, in time $\mathcal{O}(nm(kn + m))$ we can construct an equivalent instance of PROPER INTERVAL COMPLETION with $\mathcal{O}(k^3)$ vertices.

The algorithm of Theorem 1.1 starts with applying the kernelization algorithm of Theorem 2.5. This step contributes $\mathcal{O}(nm(kn + m))$ to the running time, and all further computation will take $k^{\mathcal{O}(k^{2/3})}$ time, yielding the promised time bound.

In case of a positive answer, our algorithm also finds a canonical ordering of the (reduced, kernelized) instance. We remark here that, although it is not stated explicitly in the work of Bessy and Perez [2], all the reduction rules of [2] are straightforward to reverse. More precisely, apart from Rule 2.3 of [2], which just greedily adds some edges to the solution, all other reduction rules of [2] remove a vertex from the input graph, providing (in the proof of safeness) a way of inserting it back after a solution to the reduced instance is found. A direct implementation of the aforementioned method of inserting back a single deleted vertex works in time $\mathcal{O}(nm)$, allowing us to lift a solution of the reduced instance to the original one in $\mathcal{O}(n^2m)$ total time.

Hence, in the rest of the paper we may assume that we are given a PROPER INTERVAL COMPLETION instance (G, k) with $n = |V(G)| = \mathcal{O}(k^3)$, and we are targeting the canonical umbrella ordering of G provided that it yields a completion of size at most k . Moreover, we assume that G is connected, as we may otherwise solve each connected component of G independently, determining in each component the size of minimum possible solution.

Lexicographically minimum perfect matching. In a few places we need the following greedy procedure to find some canonical object.

LEMMA 2.6. Given two linearly ordered sets $X = \{x_1 \prec x_2 \prec \dots \prec x_s\}$ and $Y = \{y_1 \prec y_2 \prec \dots \prec y_s\}$, and allowed sets $A_i \subseteq Y$ for each $1 \leq i \leq s$, one can in polynomial time either find a bijection $f : X \rightarrow Y$ that satisfies

$$(2.1) \quad f(x_i) \in A_i \quad \text{for any } 1 \leq i \leq s$$

and, subject to (2.1), yields lexicographically minimum sequence $(f(x_1), f(x_2), \dots, f(x_s))$, or correctly conclude that such a bijection does not exist.

Proof. We model the task of satisfying the condition (2.1) as a problem of finding a perfect matching in a bipartite graph, which can be solved in polynomial time. We construct an auxiliary bipartite graph H with bipartition classes X and Y and make each $x_i \in X$ adjacent to all $y_j \in A_i$. Clearly, any perfect matching in H corresponds to a bijection f satisfying (2.1).

To obtain the lexicographically minimum sequence $(f(x_1), f(x_2), \dots, f(x_s))$, we use the self-reducibility of the task of finding a perfect matching. That is, for each $i = 1, 2, \dots, s$ we try to match x_i . When we consider x_i , we try each $j = 1, 2, \dots, s$ and, whenever y_j is yet unmatched and $y_j \in A_i$, we temporarily match x_i with y_j and compute whether the subgraph induced by the currently unmatched vertices contains a perfect matching. If this is true, we fix the match $f(x_i) = y_j$, and otherwise we proceed to the next vertex y_j . It is straightforward to verify that this procedure indeed yields f as desired. \square

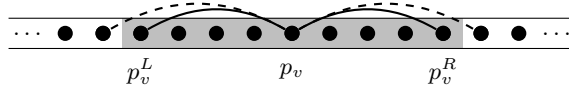


FIG. 4. The definition of values p_v , p_v^L , and p_v^R for an expensive vertex v . The gray area denotes $N_{G^\sigma}[v]$.

3. Expensive vertices. Recall that we are given a PROPER INTERVAL COMPLETION instance (G, k) and we want to reason about its canonical umbrella ordering, denoted σ , provided that (G, k) is a YES-instance. In this section we deal with vertices that are incident with many edges of F^σ . Formally, we set a threshold $\tau := (2k)^{1/3}$ and say that a vertex v is *expensive* with respect to σ if $|F^\sigma(v)| > \tau$ and *cheap* otherwise. Note that there are at most $(2k)^{2/3} = \tau^2$ expensive vertices, and given that $|V(G)|$ is bounded polynomially in k , we may afford guessing a lot of information about expensive vertices within the promised time bound. Our goal is to get rid of expensive vertices, at the cost of turning our PROPER INTERVAL COMPLETION instance (G, k) into a SPIC instance.

More formally, we branch into $k^{\mathcal{O}(k/\tau)} = k^{\mathcal{O}(k^{2/3})}$ subcases, considering all possible values for the following (see also Figure 4):

1. a set $V_\S \subseteq V(G)$ of all expensive vertices with respect to σ ,
2. for every $v \in V_\S$, integers p_v , p_v^L , and p_v^R satisfying $p_v = \sigma(v)$, $p_v^L = \min\{\sigma(w) : w \in N_{G^\sigma}[v]\}$ and $p_v^R = \max\{\sigma(w) : w \in N_{G^\sigma}[v]\}$.

In each branch, we look for the canonical minimum solution to the instance (G, k) , assuming that the aforementioned guess is a correct one. The *correct branch* is the one where this assumption is indeed true.

We now perform some cleanup operations. First, observe that from the definition of an umbrella ordering it follows that in the correct branch $w \in N_{G^\sigma}[v]$ if and only if $p_v^L \leq \sigma(w) \leq p_v^R$. In particular, $p_v^L \leq p_v \leq p_v^R$. Consider now a pair $v_1, v_2 \in V_\S$ and observe the following. If $p_{v_1} \leq p_{v_2}$, then the properties of an umbrella ordering imply that $p_{v_1}^L \leq p_{v_2}^L$ and $p_{v_1}^R \leq p_{v_2}^R$. Hence, we terminate all the branches where any of these inequalities is not satisfied, or where $p_{v_1} = p_{v_2}$ for some $v_1 \neq v_2$.

Furthermore, note that in the correct branch we have $v_1 v_2 \in E(G^\sigma)$ if and only if $p_{v_2} \in [p_{v_1}^L, p_{v_1}^R]$ and $p_{v_1} \in [p_{v_2}^L, p_{v_2}^R]$, and $v_1 v_2 \notin E(G^\sigma)$ if and only if *neither* of the two aforementioned inclusions holds. Thus, we terminate the branch if *exactly one* of these inclusions holds, or if $v_1 v_2 \in E(G)$ and at least one of them does not hold.

Denote $\Sigma_\S = \{p_v : v \in V_\S\}$ as the set of positions guessed to be used by the expensive vertices and $\Sigma = \{1, 2, \dots, n\} \setminus \Sigma_\S$ as the set of the remaining positions. For every $1 \leq i \leq |\Sigma|$, by $\pi(i)$ we denote the i th position of Σ . Define also $\sigma_\S : V_\S \rightarrow \Sigma_\S$ as $\sigma_\S(v) = p_v$.

We compute a set F_\S consisting of all (unordered) pairs $v_1, v_2 \in V_\S$ such that $v_1 v_2 \notin E(G)$, but $p_{v_2} \in [p_{v_1}^L, p_{v_1}^R]$, that is, the guessed values imply that $v_1 v_2 \in E(G^\sigma)$ and, consequently, $F_\S = F^\sigma \cap \binom{V_\S}{2}$ in the correct branch. Observe the following.

LEMMA 3.1. *In all branches F_\S is a completion of $G[V_\S]$, and σ_\S , treated as an ordering of V_\S , is an umbrella ordering of $G[V_\S] + F_\S$.*

Proof. Consider any $a, b, c \in V_\S$ with $\sigma_\S(a) < \sigma_\S(b) < \sigma_\S(c)$. If $ac \in E(G) \cup F_\S$, then it follows from the cleanup operations and the definition of F_\S that $\sigma_\S(c) \in [p_a^L, p_a^R]$ and $\sigma_\S(a) \in [p_c^L, p_c^R]$. Recall that $\sigma_\S(a) \in [p_a^L, p_a^R]$ and $\sigma_\S(c) \in [p_c^L, p_c^R]$. Hence, $\sigma_\S(b) \in [\sigma_\S(a), \sigma_\S(c)] \subseteq [p_a^L, p_a^R] \cap [p_c^L, p_c^R]$ and $ab, bc \in E(G) \cup F_\S$. \square

Consider now a vertex $u \notin V_{\S}$. For any $v \in V_{\S}$, if $uv \in E(G)$, then in the correct branch $\sigma(u) \in [p_v^L, p_v^R]$. This motivates us to define

$$\Sigma_u = \pi^{-1} \left(\Sigma \cap \bigcap_{v \in V_{\S} \cap N_G(u)} [p_v^L, p_v^R] \right).$$

Observe that in the correct branch $\pi^{-1}(\sigma(u)) \in \Sigma_u$.

Furthermore, observe that, in the correct branch, if $uv \notin E(G)$ for some $u \notin V_{\S}$ and $v \in V_{\S}$, then exactly one of the following holds: $uv \in F^\sigma$ or $\sigma(u) \notin [p_v^L, p_v^R]$. In other words, a vertex $v \in V_{\S}$ has degree exactly $p_v^R - p_v^L$ in the graph G^σ . This motivates us to define the following cost value for every branch:

$$c_{\S} = -|F_{\S}| + \sum_{v \in V_{\S}} ((p_v^R - p_v^L) - \deg_G(v)).$$

Observe that this cost function is actually meaningful for every branch.

LEMMA 3.2. *Let σ' be an ordering of $V(G)$ and F be a completion of G such that*

- (i) *σ' is an umbrella ordering of $G + F$ and*
- (ii) *for every $v \in V_{\S}$ we have $\sigma'(v) = p_v$ and $\sigma'(N_{G+F}[v]) = [p_v^L, p_v^R]$. Then there are exactly c_{\S} edges of F that are incident with V_{\S} .*

Proof. Observe that the degree of $v \in V_{\S}$ in $G + F$ is exactly $p_v^R - p_v^L$. Hence, exactly $p_v^R - p_v^L - \deg_G(v)$ edges of F are incident with v and the sum $\sum_{v \in V_{\S}} ((p_v^R - p_v^L) - \deg_G(v))$ counts the edges of F incident with V_{\S} but double-counts the edges of F with both endpoints in V_{\S} . However, the set of double-counted edges is exactly $F \cap \binom{V_{\S}}{2} = F_{\S}$. The lemma follows. \square

We define graphs G_{\downarrow} and G_{\uparrow} with vertex set $\{1, 2, \dots, |\Sigma|\}$ as follows. For $1 \leq i < j \leq |\Sigma|$, we set $ij \in E(G_{\downarrow})$ if and only if there is a *witness* vertex $x \in V_{\S}$ such that either $p_x^L \leq \pi(i) < \pi(j) < p_x$ or $p_x < \pi(i) < \pi(j) \leq p_x^R$. For G_{\uparrow} , we set $ij \notin E(G_{\uparrow})$ if and only if there exists a *witness* vertex $y \in V_{\S}$ such that either $\pi(i) < p_y^L \leq p_y < \pi(j)$ or $\pi(i) < p_y \leq p_y^R < \pi(j)$.

The next lemma shows that G_{\downarrow} and G_{\uparrow} satisfy the requirements for being a part of a SPIC instance.

LEMMA 3.3. *Both G_{\downarrow} and G_{\uparrow} are proper interval graphs and the identity is an umbrella ordering of both of them. Moreover, in the correct branch $E(G_{\downarrow}) \subseteq E(\pi^{-1}((\sigma(G^\sigma))[\Sigma])) \subseteq E(G_{\uparrow})$.*

Proof. For the first claim, observe that in the case of G_{\downarrow} , for every edge $ij \in E(G_{\downarrow})$ with $i < j$, its witness x also witnesses that $i'j' \in E(G_{\downarrow})$ for every $i \leq i' < j' \leq j$. Similarly, in the case of G_{\uparrow} , for any nonedge $ij \notin E(G_{\uparrow})$ with $i < j$, its witness y also witnesses that $i'j' \notin E(G_{\uparrow})$ for each $i' \leq i < j \leq j'$.

We now move to the second claim, so assume we are in the correct branch. For G_{\downarrow} , observe that if $ij \in E(G_{\downarrow})$, then $\sigma^{-1}(\pi(i))\sigma^{-1}(\pi(j)) \in E(G^\sigma)$ by the umbrella property as $\sigma^{-1}(p_x^L)\sigma^{-1}(p_x) \in E(G^\sigma)$ and $\sigma^{-1}(p_x)\sigma^{-1}(p_x^R) \in E(G^\sigma)$. For G_{\uparrow} , if i, j are such that $\sigma^{-1}(\pi(i))\sigma^{-1}(\pi(j)) \in E(G^\sigma)$ and $\pi(i) < p_y < \pi(j)$ for some $y \in V_{\S}$, then by the umbrella property we have that $y\sigma^{-1}(\pi(i)), y\sigma^{-1}(\pi(j)) \in E(G^\sigma)$ and consequently $p_y^L \leq \pi(i) < p_y < \pi(j) \leq p_y^R$. Since y was chosen arbitrarily, it follows that $ij \in E(G_{\uparrow})$ and the lemma follows. \square

By Lemma 3.3, we may terminate the branches where G_{\downarrow} is not a subgraph of G_{\uparrow} .

Define $W = V(G) \setminus V_{\S}$, $H = G[W]$ and $\ell = k - c_{\S}$. Recall that in the remaining branches $\mathcal{I} := (H, \ell, (\Sigma_u)_{u \in V(G)}, G_{\downarrow}, G_{\uparrow})$ is a valid SPIC instance. In the next lemmata we show that it is sufficient to solve it instead of (G, k) .

LEMMA 3.4. *If (G, k) is a YES-instance to PROPER INTERVAL COMPLETION with the canonical umbrella ordering σ , then in the correct branch the function $\sigma_H := \pi^{-1} \circ \sigma|_W$ is a feasible ordering of the SPIC instance \mathcal{I} with $F^{\sigma_H} \subseteq F^\sigma \cap \binom{W}{2} =: F_W$; in particular, for any $u \in W$ we have $|F^{\sigma_H}(u)| \leq \tau$. Moreover, $c(\sigma_H) = |F^\sigma| - c_\S - |F_W \setminus F^{\sigma_H}| \leq |F^\sigma| - c_\S$.*

Proof. Observe that σ_H is indeed an ordering of W . We first verify that it is feasible. Clearly, in the correct branch $\sigma_H(u) = \pi^{-1}(\sigma(u)) \in \Sigma_u$ for any $u \in W$. Consider any pair u, v with $\sigma_H(u) < \sigma_H(v)$ and $\sigma_H(u)\sigma_H(v) \notin E(G_\uparrow)$. Let y be a witness that $\sigma_H(u)\sigma_H(v) \notin E(G_\uparrow)$. If $\sigma_H(u) < p_y^L \leq p_y < \sigma_H(v)$, then $uy \notin E(G^\sigma)$ and, by the umbrella property, $uv \notin E(G^\sigma)$, so in particular $uv \notin E(G)$. Symmetrically, if $\sigma_H(u) < p_y \leq p_y^R < \sigma_H(v)$, then $yv \notin E(G^\sigma)$ and, by the umbrella property, $uv \notin E(G^\sigma)$, so in particular $uv \notin E(G)$. Consequently, $uv \notin E(G)$ in both cases and σ_H is feasible.

We now show that $F^{\sigma_H} \subseteq F_W$. Consider any $uv \in F^{\sigma_H}$ and without loss of generality assume $\sigma_H(u) < \sigma_H(v)$. If there exist $u', v' \in W$ with $\sigma_H(u') \leq \sigma_H(u) < \sigma_H(v) \leq \sigma_H(v')$ and $u'v' \in E(G)$, then $\sigma(u') \leq \sigma(u) < \sigma(v) \leq \sigma(v')$ by the monotonicity of π and hence $uv \in F^\sigma$. Otherwise, by the definition of F^{σ_H} , we have that $\sigma_H(u)\sigma_H(v) \in E(G_\downarrow)$. By the definition of G_\downarrow , there exists $x \in V_\S$ with $p_x^L \leq \pi(\sigma_H(u)) = \sigma(u) < \sigma(v) = \pi(\sigma_H(v)) < p_x$ or $p_x < \pi(\sigma_H(u)) = \sigma(u) < \sigma(v) = \pi(\sigma_H(v)) \leq p_x^R$. In the first case, by the umbrella property we have that $uv \in F^\sigma$ because $\sigma^{-1}(p_x^L)x \in E(G^\sigma)$. Similarly, in the second case, $uv \in F^\sigma$ since $\sigma^{-1}(p_x^R)x \in E(G^\sigma)$.

We now compute the cost of σ_H . By Lemma 3.2, there are exactly c_\S edges of F^σ incident with V_\S . Therefore $|F_W| = |F^\sigma| - c_\S$. The already proven inclusion $F^{\sigma_H} \subseteq F_W$ finishes the proof of the formula for the cost of σ_H . \square

LEMMA 3.5. *Let σ_H be a feasible ordering of the SPIC instance \mathcal{I} in some branch. Let also σ' be an ordering of $V(G)$ such that $\sigma'(u) = \pi(\sigma_H(u))$ for $u \in W$ and $\sigma'(u) = \sigma_\S(u)$ for $u \notin W$. Then $|F^{\sigma'}| \leq c(\sigma_H) + c_\S$.*

Proof. We define

$$F = F^{\sigma_H} \cup F_\S \cup \{uv : u \in W \wedge v \in V_\S \wedge uv \notin E(G) \wedge \pi(\sigma_H(u)) \in [p_v^L, p_v^R]\}.$$

We now show that σ' is an umbrella ordering of $G + F$. Observe that if this is true, then Lemma 3.2 will yield that $|F^{\sigma'}| \leq |F| = |F^{\sigma_H}| + c_\S$, finishing the proof of the lemma; the condition (ii) of Lemma 3.2 can be directly checked from the definitions of σ', F .

Consider then a triple $a, b, c \in V(G)$ with $\sigma'(a) < \sigma'(b) < \sigma'(c)$ and $ac \in E(G) \cup F$. We consider a few cases, depending on the intersection $V_\S \cap \{a, b, c\}$.

First, consider the case $a, c \in V_\S$. If $b \in V_\S$, then $ab, bc \in E(G) \cup F$ by Lemma 3.1. Otherwise, observe that the cleanup operations imply that $\sigma'(a) = p_a \in [p_c^L, p_c^R]$ and $\sigma'(c) = p_c \in [p_a^L, p_a^R]$ and we obtain $\sigma'(b) = \pi(\sigma_H(b)) \in [p_a^L, p_a^R] \cap [p_c^L, p_c^R]$. Hence $ab, bc \in E(G) \cup F$ directly from the definition of F .

Second, consider the case $a \in V_\S$ and $c \in W$. We claim that $ac \in E(G) \cup F$ implies that $\sigma'(c) = \pi(\sigma_H(c)) \in [p_a^L, p_a^R]$. Indeed, if $ac \in F$, then this follows directly from the definition of F . If $ac \in E(G)$, however, then $\sigma'(c) = \pi(\sigma_H(c)) \in \pi(\Sigma_c) \subseteq [p_a^L, p_a^R]$ since σ_H is feasible. Now observe that since $\sigma'(a) = p_a \in [p_a^L, p_a^R]$, then we have also that $\sigma'(b) \in [p_a^L, p_a^R]$. Since $\sigma'(a) < \sigma'(b) < \sigma'(c)$, then in fact $\sigma'(b), \sigma'(c) \in [p_a, p_a^R]$.

Assume first that $b \in V_\S$. Then $ab \in E(G) \cup F_\S$ by the definition of F_\S . Moreover, as $\sigma'(b) = p_b > \sigma'(a) = p_a$, by the cleanup operations we have that $p_b^R \geq p_a^R$ and, consequently, $\sigma'(c) = \pi(\sigma_H(c)) \in [p_b^L, p_b^R]$. Hence, in this case $bc \in E(G) \cup F$ by the definition of F .

Assume now $b \in W$. Clearly $\sigma'(b) \in [p_a^L, p_a^R]$ implies that $ab \in E(G) \cup F$ by the definition of F . Moreover, observe that as both $\sigma'(b) = \pi(\sigma_H(b))$ and $\sigma'(c) = \pi(\sigma_H(c))$ belong to $[p_a, p_a^R]$, we have $\sigma_H(b)\sigma_H(c) \in G_\downarrow$ and hence $bc \in E(G) \cup F^{\sigma_H}$.

Third, observe that the case $a \in W$ and $c \in V_\S$ is symmetrical to the previous one.

Finally, consider the case $a, c \in W$, so $ac \in E(G) \cup F^{\sigma_H}$. If $b \in W$, then $ab, bc \in E(G) \cup F^{\sigma_H}$ as σ_H is an umbrella ordering of $G[W] + F^{\sigma_H}$. Hence, assume $b \in V_\S$. Observe that $ac \in E(G) \cup F^{\sigma_H}$ implies that $ac \in E(G_\uparrow)$. However, we have that $\pi(\sigma_H(a)) < p_b < \pi(\sigma_H(c))$. Thus, by the definition of G_\uparrow , we have $p_b^L \leq \pi(\sigma_H(a)) < \pi(\sigma_H(c)) \leq p_b^R$ and, by the definition of F , $ab, bc \in E(G) \cup F$. This concludes the proof of the lemma. \square

LEMMA 3.6. *If (G, k) is a YES-instance to PROPER INTERVAL COMPLETION with the canonical umbrella ordering σ , then in the correct branch the function $\sigma_H := \pi^{-1} \circ \sigma|_W$ is the canonical umbrella ordering of the SPIC instance \mathcal{I} of cost at most ℓ . Moreover, $F^{\sigma_H} = F^\sigma \cap \binom{W}{2}$; in particular, for any $u \in W$ we have $|F^{\sigma_H}(u)| \leq \tau$.*

Proof. We focus on the correct branch. By Lemma 3.4, there exists a feasible ordering of the SPIC instance \mathcal{I} . Let σ'_H be the canonical ordering of this instance. Define σ' as in Lemma 3.5 for the ordering σ'_H .

By Lemma 3.5 and the optimality of σ , we have that

$$|F^\sigma| \leq |F^{\sigma'}| \leq c(\sigma'_H) + c_\S.$$

On the other hand, by Lemma 3.4 and the optimality of σ'_H , we have that

$$c(\sigma'_H) \leq c(\pi^{-1} \circ \sigma|_W) \leq |F^\sigma| - c_\S.$$

Hence, all aforementioned inequalities are in fact equalities, and $F^{\sigma_H} = F^\sigma \cap \binom{W}{2}$. In particular, $F^{\sigma'}$ is a minimum completion of G and $\pi^{-1} \circ \sigma|_W$ is of minimum possible cost. By the monotonicity of π , we infer that the lexicographical minimization in fact chooses $\sigma'_H = \sigma_H$ and the lemma is proven. \square

In the next sections we will show the following.

THEOREM 3.7. *There exists an algorithm that, given a branch with a SPIC instance \mathcal{I} , runs in time $n^{\mathcal{O}(\ell/\tau + \tau^2)}$ and, if given the correct branch, computes the canonical ordering of \mathcal{I} .*

The equivalence shown in Lemmata 3.4, 3.5, and 3.6, together with the bound $n = \mathcal{O}(k^3)$, allows us to solve the PROPER INTERVAL COMPLETION instance (G, k) by applying the algorithm of Theorem 3.7 to each branch separately. Observe that we have $k^{\mathcal{O}(k^{2/3})}$ branches, and for $\tau = (2k)^{1/3}$, $\ell \leq k$, and $n = \mathcal{O}(k^3)$ we have $n^{\mathcal{O}(\ell/\tau + \tau^2)} = k^{\mathcal{O}(k^{2/3})}$; therefore, the running time will be as guaranteed in Theorem 1.1.

Hence, it remains to prove Theorem 3.7. In its proof it will be clear that the algorithm runs within the given time bound. Hence, we assume that we work in the correct branch and we will mostly focus on proving that we indeed find the canonical ordering of \mathcal{I} .

4. Sections. We now proceed with the proof of Theorem 3.7. Assume we are given the correct branch with a SPIC instance $\mathcal{I} = (H, \ell, (\Sigma_u)_{u \in V(G)}, G_\downarrow, G_\uparrow)$. Recall that we look for the canonical ordering σ_H of \mathcal{I} and we assume that σ_H is of cost at most ℓ and $|F^{\sigma_H}(u)| \leq \tau$ for every $u \in V(H)$. The last assumption allows us to guess

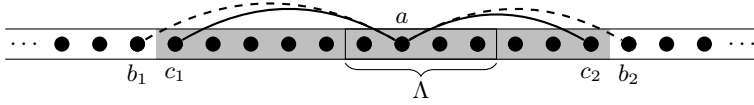


FIG. 5. The guessed vertices $a, b_1, b_2, c_1,$ and c_2 with respect to a twin class Λ . The gray area denotes $N_{H^{\sigma_H}}(\Lambda)$.

edges $F^{\sigma_H}(u)$ for a set of carefully chosen vertices $u \in V(H)$. In this section we use this property to show the following statement.

DEFINITION 4.1. A section is a subset A of $V(H)$. A section A is consistent with an ordering σ_H if σ_H maps A onto the first $|A|$ positions.

THEOREM 4.2. In $k^{\mathcal{O}(\tau)}$ time one can enumerate a family \mathcal{S} of $k^{\mathcal{O}(\tau)}$ sections that contains all sections consistent with the canonical ordering σ_H .

The proof of Theorem 4.2 is divided into two steps. First, we investigate true twin classes in the graph H^{σ_H} and show that we can efficiently enumerate a small family of candidates for these twin classes. Then we use the twin class residing at position $|A| + 1$ to efficiently “guess” a section A consistent with the canonical ordering σ_H . Henceforth we assume that the canonical ordering σ_H is of cost at most k .

4.1. Potential twin classes. Recall that two vertices x and y are true twins if $N[x] = N[y]$; in particular, this implies that they are adjacent. The relation of being a true twin is an equivalence relation, and an equivalence class of this relation is called a twin class. We make the following observation, straightforward from the definition of an umbrella ordering.

LEMMA 4.3. In an umbrella ordering of a proper interval graph, the vertices of any twin class occupy consecutive positions.

The main result of this section is the following.

THEOREM 4.4. In $k^{\mathcal{O}(\tau)}$ time one can enumerate a family \mathcal{T} of $k^{\mathcal{O}(\tau)}$ triples $(L, \Lambda, \sigma_\Lambda)$ such that for any twin class Λ of H^{σ_H} , if L is the set of vertices of H placed to the left of Λ in the ordering σ_H , then $(L, \Lambda, \sigma_H|_\Lambda) \in \mathcal{T}$.

We describe the algorithm of Theorem 4.4 as a branching algorithm that produces $k^{\mathcal{O}(\tau)}$ subcases and, in each subcase, produces one tuple $(L, \Lambda, \sigma_\Lambda)$. We fix one twin class Λ of H^{σ_H} and argue that the algorithm in one of the branches produces $(L, \Lambda, \sigma_H|_\Lambda)$, where L is defined as in Theorem 4.4. We perform this task in two phases: we first reason about L and Λ , and then we deduce the ordering $\sigma_H|_\Lambda$.

4.1.1. Phase one: L and Λ . The algorithm guesses the following five vertices (see also Figure 5):

1. a is any vertex of Λ ;
2. b_1 is the rightmost vertex outside $N_{H^{\sigma_H}}[\Lambda]$ in σ_H that lies before Λ , or $b_1 = \perp$ if no such vertex exists;
3. c_1 is the leftmost vertex of $N_{H^{\sigma_H}}[\Lambda]$ in σ_H ;
4. c_2 is the rightmost vertex of $N_{H^{\sigma_H}}[\Lambda]$ in σ_H ;
5. b_2 is the leftmost vertex outside $N_{H^{\sigma_H}}[\Lambda]$ in σ_H that lies after Λ , or $b_2 = \perp$ if no such vertex exists.

Moreover, for each $u \in \{a, b_1, b_2, c_1, c_2\} \setminus \{\perp\}$ the algorithm guesses $F^{\sigma_H}(u)$. This leads us to $k^{\mathcal{O}(\tau)}$ subcases. We now argue that if the guesses are correct, we can deduce the pair (L, Λ) . The crucial step is the following.

LEMMA 4.5. In the branch where the guesses are correct, the following holds for any $u \in N_{H^{\sigma_H}}[a]$:

1. if $u \in N_{H^{\sigma_H}}[b_1]$ or $u \notin N_{H^{\sigma_H}}[c_2]$, then $u \notin \Lambda$ and u lies before Λ in the ordering σ_H ;
2. if $u \in N_{H^{\sigma_H}}[b_2]$ or $u \notin N_{H^{\sigma_H}}[c_1]$, then $u \notin \Lambda$ and u lies after Λ in the ordering σ_H ;
3. if none of the above happens, then $u \in \Lambda$.

Here we take the convention that $N_{H^{\sigma_H}}[\perp] = \emptyset$.

Proof. By the definition of $b_1, b_2, c_1,$ and c_2 , we have that every vertex $u \in \Lambda$ lies in $N_{H^{\sigma_H}}[c_1]$ and $N_{H^{\sigma_H}}[c_2]$, but not in $N_{H^{\sigma_H}}[b_1]$ nor in $N_{H^{\sigma_H}}[b_2]$. Consequently, any vertex of Λ falls into the third category of the statement of the lemma.

We now show that any other vertex of $N_{H^{\sigma_H}}[a]$ falls into one of the first two categories, depending on its position in the ordering σ_H . By symmetry, we may only consider a vertex $u \in N_{H^{\sigma_H}}[a] \setminus \Lambda$ that lies before Λ in σ_H . Note that the umbrella property together with $a \notin N_{H^{\sigma_H}}[b_2]$ implies that $u \notin N_{H^{\sigma_H}}[b_2]$ and together with $ac_1 \in E(H^{\sigma_H})$ implies $uc_1 \in E(H^{\sigma_H})$. Consequently, u does not fall into the second category in the statement of the lemma. We now show that it falls into the first one.

As $u \notin \Lambda$ and $u \in N_{H^{\sigma_H}}[a]$, either $N_{H^{\sigma_H}}(u) \setminus N_{H^{\sigma_H}}[a]$ is not empty or $N_{H^{\sigma_H}}(a) \setminus N_{H^{\sigma_H}}[u]$ is not empty. In the first case, let $uw \in E(H^{\sigma_H})$ but $aw \notin E(H^{\sigma_H})$. Since also $ua \in E(H^{\sigma_H})$, by the umbrella property it easily follows that w lies before u in the ordering σ_H , so in particular before Λ . By the definition of b_1 , b_1 exists and $\sigma_H(b_1) \geq \sigma_H(w)$. By the umbrella property, $b_1u \in E(H^{\sigma_H})$ and hence $u \in N_{H^{\sigma_H}}[b_1]$.

In the second case, assume $uw \notin E(H^{\sigma_H})$ but $aw \in E(H^{\sigma_H})$. Again, since $ua \in E(H^{\sigma_H})$, by the umbrella property it easily follows that w lies after Λ in the ordering σ_H , so in particular after u . By the definition of c_2 and the existence of w , $c_2 \notin \Lambda$ and $\sigma_H(c_2) \geq \sigma_H(w)$. By the umbrella property, $c_2u \notin E(H^{\sigma_H})$ and $u \notin N_{H^{\sigma_H}}[c_2]$. Hence, u falls into the first category and the lemma is proven. \square

The knowledge of a and $F^{\sigma_H}(a)$ allows us to compute $N_{H^{\sigma_H}}[\Lambda] = N_{H^{\sigma_H}}[a]$. By making use of Lemma 4.5, we can further partition $N_{H^{\sigma_H}}[\Lambda]$ into Λ , the vertices of $N_{H^{\sigma_H}}(\Lambda)$ that lie before Λ in the ordering σ_H , and the ones that lie after Λ . We are left with the vertices outside $N_{H^{\sigma_H}}[\Lambda]$.

We guess the position i such that the first vertex of Λ in the ordering σ_H is in position i . Note that, by Lemma 4.3, the vertices of Λ occupy positions $i, i + 1, \dots, i + |\Lambda| - 1$ in σ_H .

Let C be a connected component of $H \setminus N_{H^{\sigma_H}}[\Lambda]$. Recall that by Lemma 3.6, $\sigma_H = \pi^{-1} \circ \sigma|_W$ and $F^{\sigma_H} = F^\sigma \cap \binom{W}{2}$. As no vertex of C is incident with Λ in H^{σ_H} , by the properties of an umbrella ordering we infer that all vertices of $N_G[C]$ lie before position $\pi(i)$ or all vertices of $N_G[C]$ lie after position $\pi(i + |\Lambda| - 1)$ in the ordering $\pi \circ \sigma_H = \sigma|_W$. As G is assumed to be connected, $N_G(C)$ contains a vertex of $N_{H^{\sigma_H}}[\Lambda]$ or of $V_{\mathfrak{s}}$. Any such vertex allows us to deduce which of the two aforementioned options is true for C in σ . This allows us to decide whether $C \subseteq L$ or $L \cap C = \emptyset$, and consequently deduce the set L . Note that it must hold that $|L| = i - 1$, and otherwise we may discard the guess.

4.1.2. Phase two: The ordering $\sigma_H|_\Lambda$. We are left with determining $\sigma_H|_\Lambda$. Note that we already know the domain Λ and the codomain $\{i, i + 1, \dots, i + |\Lambda| - 1\}$ of this bijection. We prove the following.

LEMMA 4.6. *The bijection $\sigma_H|_\Lambda$ is the lexicographically minimum bijection $\sigma_\Lambda : \Lambda \rightarrow \{i, i + 1, \dots, i + |\Lambda| - 1\}$ among those bijections σ_Λ that satisfy $\sigma_\Lambda(u) \in \Sigma_u$ for any $u \in \Lambda$.*

Proof. Let $\sigma_\Lambda : \Lambda \rightarrow \{i, i + 1, \dots, i + |\Lambda| - 1\}$ be the lexicographically minimum bijection among those that satisfy $\sigma_\Lambda(u) \in \Sigma_u$ for any $u \in \Lambda$; note that at least one

such bijection exists, since $\sigma_H|_\Lambda$ is one. Consider an ordering σ' of $V(H)$ defined as follows: $\sigma'(u) = \sigma_\Lambda(u)$ if $u \in \Lambda$ and $\sigma'(u) = \sigma_H(u)$ otherwise. Observe that σ' is an ordering of $V(H)$. Moreover, as Λ is a twin class of H^{σ_H} , we have $\sigma'(H^{\sigma_H}) = \sigma(H^{\sigma_H})$. Hence σ' is a feasible ordering of H and umbrella ordering of H^{σ_H} . We infer that $F^{\sigma'} \subseteq F^{\sigma_H}$. On the other hand, as σ_H is the canonical solution, we have $c(\sigma_H) \leq c(\sigma')$. Hence, both aforementioned inequalities are in fact tight and $F^{\sigma'} = F^{\sigma_H}$. Furthermore, the lexicographical minimization criterion implies that $\sigma_\Lambda = \sigma_H|_\Lambda$ and $\sigma' = \sigma_H$. \square

Finally, observe that the characterization of $\sigma_H|_\Lambda$ given by Lemma 4.6 fits into the conditions of Lemma 2.6 and, consequently, $\sigma_H|_\Lambda$ can be computed in polynomial time given L, Λ , and the index i . This concludes the proof of Theorem 4.4.

4.2. Proof of Theorem 4.2. Given Theorem 4.4, the proof of Theorem 4.2 is now straightforward. We first compute the family \mathcal{T} of Theorem 4.4. Then, for each $(L, \Lambda, \sigma_\Lambda) \in \mathcal{T}$ and each position $p \in \{1, 2, \dots, |V(H)|\}$ we output a set

$$A := L \cup \{u \in \Lambda : \sigma_\Lambda(u) < p\}.$$

Additionally, we output a section $V(H)$. Clearly, the algorithm outputs $k^{\mathcal{O}(\tau)}$ sections and works within the promised time bound. It remains to argue that it outputs all sections consistent with σ_H .

Consider a section A consistent with σ_H , that is, $A = \sigma^{-1}(\{1, 2, \dots, |A|\})$. If $A = V(H)$, the statement is obvious, so assume otherwise. Consider the position $p := |A| + 1$, let $u = \sigma^{-1}(p)$ and let Λ be the twin class of u in H^{σ_H} . Moreover, let L be the set of vertices of H placed before Λ in σ_H . By Theorem 4.4, $(L, \Lambda, \sigma_H|_\Lambda) \in \mathcal{T}$. Moreover, note that the algorithm outputs exactly the set A when it considers the triple $(L, \Lambda, \sigma_H|_\Lambda)$ and position p . This concludes the proof of Theorem 4.2. \square

5. Dynamic programming. In this section we conclude the proof of Theorem 3.7 by showing the following.

THEOREM 5.1. *Given a SPIC instance $\mathcal{I} = (G, k, (\Sigma_u)_{u \in V(G)}, G_\downarrow, G_\uparrow)$ with $n = |V(G)|$, a threshold τ , and a family $\mathcal{S} \subseteq 2^{V(G)}$, one can in $n^{\mathcal{O}(k/\tau + \tau)} |\mathcal{S}|^{\mathcal{O}(\tau)}$ time find the canonical ordering σ of \mathcal{I} , assuming that*

1. $c(\sigma) \leq k$;
2. for each $u \in V(G)$, $|F^\sigma(u)| \leq \tau$;
3. each section consistent with σ belongs to \mathcal{S} .

Observe that if we apply Theorem 5.1 to a branch with a SPIC instance \mathcal{I} , the threshold τ , and family \mathcal{S} output by Theorem 4.2, then we obtain the algorithm promised by Theorem 3.7.

The algorithm of Theorem 5.1 is a dynamic programming algorithm. Henceforth assume that the instance \mathcal{I} with threshold τ and family \mathcal{S} is as promised in the statement of Theorem 5.1, and let σ be the canonical ordering of \mathcal{I} . We develop two different ways of separating the graphs G and G^σ into smaller parts, suitable for dynamic programming. Consequently, the dynamic programming algorithm has in some sense “two layers” and two different types of states.

5.1. Layer one: Jumps and jump sets. We first develop a way to split the graphs G and G^σ “vertically.” To this end, first denote for any position p the section $A_p = \{v \in V(G) : \sigma(v) < p\}$; note that this definition also makes sense for $p = \infty$ and $A_\infty = V(G)$. Second, for any position p define

$$\text{jump}(p) = \min\{q : q > p \wedge \sigma^{-1}(p)\sigma^{-1}(q) \notin E(G^\sigma)\};$$

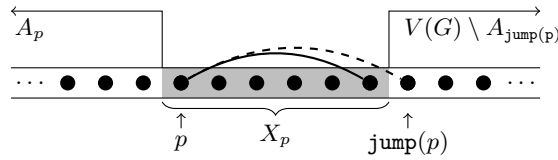


FIG. 6. A jump at position p and the corresponding jump set. The jump set X_p , denoted with gray, is a clique in G^σ , and no edge of G^σ connects A_p with $V(G) \setminus A_{\text{jump}(p)}$.



FIG. 7. Augmentation of the input graph G .

in this definition we follow the convention that the minimum of an empty set is ∞ . Moreover, we define a *jump set* for position p as

$$X_p = \sigma^{-1}([p, \text{jump}(p) - 1]) = A_{\text{jump}(p)} \setminus A_p.$$

See also Figure 6 for an illustration.

The next two lemmata follow directly from the definition of a jump and the properties of umbrella orderings.

LEMMA 5.2. *For any positions p and q , if $p \leq q$, then $\text{jump}(p) \leq \text{jump}(q)$.*

LEMMA 5.3. *Jump set X_p is a clique in G^σ , but no edge of G^σ connects a vertex of A_p with a vertex of $V(G) \setminus A_{\text{jump}(p)}$.*

We now slightly augment the graph G so that $\text{jump}(p) \neq \infty$ for all interesting positions; see also Figure 7. We take $\mathcal{O}(n^2)$ branches, guessing the first and the last vertex of G in the ordering σ , and denote them by α and ω . We introduce new vertices $\alpha_1, \alpha_2, \omega_1, \omega_2, \omega_3$ and new edges $\alpha_1\alpha_2, \alpha\alpha_1, \omega_2\omega_3, \omega_1\omega_2, \omega\omega_1$ in G . We also introduce new positions $-1, 0, n+1, n+2, n+3$, isolated in G_\downarrow and connected by edges $\{-1, 0\}, \{0, 1\}, \{n, n+1\}, \{n+1, n+2\}, \{n+2, n+3\}$ in G_\uparrow . We define $\Sigma_{\alpha_1} = \{0\}$, $\Sigma_{\alpha_2} = \{-1\}$, $\Sigma_{\omega_1} = \{n+1\}$, $\Sigma_{\omega_2} = \{n+2\}$, and $\Sigma_{\omega_3} = \{n+3\}$. Moreover, we put α_2 and α_1 before all vertices of G in the ordering σ_0 , and ω_1, ω_2 , and ω_3 after them. Note that if we precede with all the vertices in the ordering σ with α_2, α_1 and succeed with $\omega_1, \omega_2, \omega_3$ we obtain an ordering with no higher cost. Due to the way we have extended σ_0 to the new vertices, the extended ordering σ defined in this way is the canonical ordering of the extended graph G . Hence, we may abuse the notation and denote by G the graph after the addition of these five new vertices and assume that $V(G_\downarrow) = V(G_\uparrow) = \{1, 2, \dots, |V(G)|\}$ again.

Observe now that $\text{jump}(1) = 3$ and $X_1 = \{\alpha_2, \alpha_1\}$, as $\sigma^{-1}(1) = \alpha_2$ and $\sigma^{-1}(3) = \alpha$. Moreover, $\text{jump}(n-2) = n$ and $X_{n-2} = \{\omega_1, \omega_2\}$, as $\sigma^{-1}(n-2) = \omega_1$, $\sigma^{-1}(n-1) = \omega_2$, and $\sigma^{-1}(n) = \omega_3$.

The main observation now is that a jump set, together with all edges of F^σ incident with it (i.e., $F^\sigma(X_p)$) contains all sufficient information to divide the problem into parts before and after a jump set.

LEMMA 5.4. *For any position p , the following holds:*

1. For any $u_1, u_2 \in X_p$ such that $\sigma(u_1) \leq \sigma(u_2)$ we have

$$(5.1) \quad N_{G^\sigma}(u_1) \cap A_p \supseteq N_{G^\sigma}(u_2) \cap A_p,$$

$$(5.2) \quad N_{G^\sigma}(u_1) \setminus A_{\text{jump}(p)} \subseteq N_{G^\sigma}(u_2) \setminus A_{\text{jump}(p)}.$$

2. For any bijection $\sigma_p : X_p \rightarrow [p, \text{jump}(p) - 1]$ such that $\sigma_p(u) \in \Sigma_u$ for any $u \in X_p$ and both inclusions (5.1) and (5.2) hold for any $u_1, u_2 \in X_p$ with $\sigma_p(u_1) \leq \sigma_p(u_2)$, if we define an ordering σ' of $V(G)$ as $\sigma'(u) = \sigma_p(u)$ if $u \in X_p$ and $\sigma'(u) = \sigma(u)$ otherwise, then σ' is feasible and $\sigma'(G^{\sigma'})$ is a subgraph of $\sigma(G^\sigma)$.

Proof. The first statement is straightforward from the properties of an umbrella ordering. Let σ_p and σ' be as in the second statement. Observe that inclusions (5.1) and (5.2), together with the fact that X_p is a clique in $\sigma(G^\sigma)$, imply that σ' and σ differ only on the internal order of twin classes of G^σ and consequently $\sigma'(G^{\sigma'}) = \sigma(G^\sigma)$. Together with the fact that $\sigma'(u) \in \Sigma_u$ for any $u \in V(G)$, this means that σ' is a feasible ordering of G and an umbrella ordering of G^σ . Consequently $F^{\sigma'} \subseteq F^\sigma$, $\sigma'(G^{\sigma'})$ is a subgraph of $\sigma'(G^\sigma) = \sigma(G^\sigma)$, and the lemma is proven. \square

We use Lemma 5.4 to fit the task of computing $\sigma|_{X_p}$ into Lemma 2.6.

LEMMA 5.5. *Given a position p and the sets X_p, A_p and $F^\sigma(X_p)$, one can in polynomial time compute the ordering $\sigma|_{X_p}$.*

Proof. First, observe that the data promised in the lemma statement allows us to compute $N_{G^\sigma}(u) \cap A_p$ and $N_{G^\sigma}(u) \setminus A_{\text{jump}(p)}$ for every $u \in X_p$. Define a binary relation \preceq on X_p as $u_1 \preceq u_2$ if and only if both (5.1) and (5.2) hold for u_1 and u_2 . Lemma 5.4 asserts that \preceq is a total quasi-order on X_p . That is, the set X_p can be partitioned into sets U_1, U_2, \dots, U_s such that $u_1 \preceq u_2$ and $u_2 \preceq u_1$ for any $1 \leq j \leq s$ and $u_1, u_2 \in U_j$, and $u_1 \preceq u_2, u_2 \not\preceq u_1$ for any $1 \leq j_1 < j_2 \leq s$ and $u_1 \in U_{j_1}, u_2 \in U_{j_2}$. (Formally, we terminate the current branch if \preceq does not satisfy these properties.)

Observe that $\sigma|_{X_p}$ maps X_p onto $[p, \text{jump}(p) - 1]$. Lemma 5.4 asserts that all vertices of U_1 are placed by σ on the first $|U_1|$ positions of the range of $\sigma|_{X_p}$, all vertices of U_2 are placed on the next $|U_2|$ positions, etc. We use Lemma 2.6 to find a lexicographically minimum ordering σ_p that satisfies the above and additionally $\sigma_p(u) \in \Sigma_u$ for each $u \in X_p$. Define σ' as in Lemma 5.4. By the minimality of σ , we have $c(\sigma') \geq c(\sigma)$, but Lemma 5.4 asserts that $\sigma'(G^{\sigma'})$ is a subgraph of $\sigma(G^\sigma)$. Hence, σ' is of minimum possible cost. By the lexicographical minimality of σ_p , we have $\sigma_p = \sigma|_{X_p}$ and the lemma is proven. \square

With help of family \mathcal{S} , Lemma 5.5 allows us to efficiently enumerate jump sets with their surroundings.

THEOREM 5.6. *One can in $n^{\mathcal{O}(k/\tau)}|\mathcal{S}|^2$ time enumerate a family \mathcal{J} of at most $n^{\mathcal{O}(k/\tau)}|\mathcal{S}|^2$ tuples (A, X, σ_X) such that*

1. in each tuple (A, X, σ_X) we have
 - (a) $A, X \subseteq V(G)$ and $A \cap X = \emptyset$;
 - (b) $G_\downarrow[|A| + 1, |A| + |X|]$ is a complete graph,
 - (c) σ_X is a bijection between X and $[|A| + 1, |A| + |X|]$;
2. for any position p , if there are at most $2k/\tau$ edges of F^σ incident to X_p , then the tuple $J^\sigma(p) := (A_p, X_p, \sigma|_{X_p})$ belongs to \mathcal{J} .

Proof. We provide a procedure of guessing at most $n^{\mathcal{O}(k/\tau)}|\mathcal{S}|^2$ candidate tuples that will constitute the family \mathcal{J} . Since the promised properties of elements of \mathcal{J} can be checked in polynomial time, it suffices to argue that every triple of the form $(A_p, X_p, \sigma|_{X_p})$ will be among the guessed candidates.

The number of choices for A_p and $A_{\text{jump}(p)}$ is $|\mathcal{S}|^2$. Observe that then $X_p = A_{\text{jump}(p)} \setminus A_p$. Furthermore, there are $n^{\mathcal{O}(k/\tau)}$ ways to choose $F^\sigma(X_p)$ and, by Lemma 5.5, we can further deduce $\sigma|_{X_p}$. Finally, observe that by the definition of a jump it follows that every triple $(A_p, X_p, \sigma|_{X_p})$ satisfies the promised properties of the elements of \mathcal{J} . \square

We are now ready to describe the first layer of our dynamic programming algorithm.

DEFINITION 5.7 (layer-one state). *A layer-one state is a pair (J^1, J^2) of two elements of \mathcal{J} , $J^1 = (A^1, X^1, \sigma_X^1)$, $J^2 = (A^2, X^2, \sigma_X^2)$ such that $A^1 \subseteq A^2$ and $(A^1 \cup X^1) \subseteq (A^2 \cup X^2)$. The value of a layer-one state (J^1, J^2) is a bijection $f[J^1, J^2] : (A^2 \cup X^2) \setminus A^1 \rightarrow [|A^1| + 1, |A^2 \cup X^2|]$ satisfying the following:*

1. $f[J^1, J^2]$ is a feasible ordering of its domain, that is, for any $u \in (A^2 \cup X^2) \setminus A^1$ we have $f[J^1, J^2](u) \in \Sigma_u$ and for any $u_1, u_2 \in (A^2 \cup X^2) \setminus A^1$ such that $u_1 u_2 \in E(G)$, we have $f[J^1, J^2](u_1) f[J^1, J^2](u_2) \in E(G_\uparrow)$;
2. $f[J^1, J^2](u) = \sigma_X^1(u)$ for any $u \in X^1$ and $f[J^1, J^2](u) = \sigma_X^2(u)$ for any $u \in X^2$;
3. among all functions f satisfying the previous conditions, $f[J^1, J^2]$ minimizes the cardinality of F^f (where in the expression F^f the function f is treated as an ordering of the set $(A^2 \cup X^2) \setminus A^1$ in the SPIC instance $(G, k, (\Sigma_u)_{u \in V(G)}, G_\downarrow, G_\uparrow)$);
4. among all functions f satisfying the previous conditions, $f[J^1, J^2]$ is lexicographically minimum.

We first observe the following consequence of the above definition.

LEMMA 5.8. *For any $p^1 \leq p^2$ such that $J^\sigma(p^1), J^\sigma(p^2) \in \mathcal{J}$, we have that $(J^\sigma(p^1), J^\sigma(p^2))$ is a layer-one state and*

$$f[J^\sigma(p^1), J^\sigma(p^2)] = \sigma|_{A_{\text{jump}(p^2)} \setminus A_{p^1}}.$$

In particular, $\sigma = f[J^\sigma(1), J^\sigma(n-2)] \cup \{(\omega_3, n)\}$.

Proof. Let $M := A_{\text{jump}(p^2)} \setminus A_{p^1}$. It is straightforward to verify that $(J^\sigma(p^1), J^\sigma(p^2))$ is a layer-one state and $\sigma|_M$ satisfies the first two properties of the value of a layer-one state. Also, no edges of F^σ are incident to X_1 nor to X_{n-2} , and hence $J^\sigma(1), J^\sigma(n-2) \in \mathcal{J}$ and $(J^\sigma(1), J^\sigma(n-2))$ is a layer-one state.

Let f be any function satisfying the first three conditions of the definition of a value of the layer-one state $(J^\sigma(p^1), J^\sigma(p^2))$. Let σ' be an ordering of $V(G)$ defined as $\sigma'(u) = f(u)$ if u is the domain of f , and $\sigma'(u) = \sigma(u)$ otherwise. It is straightforward to verify that σ' is feasible, using the separation property provided by Lemma 5.3 and the fact that $\sigma'|_{X^1 \cup X^2} = \sigma|_{X^1 \cup X^2}$. For the same reasons, by the definition of σ' we have that $F^{\sigma'} = (F^\sigma \setminus \binom{M}{2}) \cup F^f$. By the optimality of f we have that $|F^f| \leq |F^{\sigma|M}| \leq |F^\sigma \cap \binom{M}{2}|$, and so $|F^{\sigma'}| \leq |F^\sigma|$. By the optimality of σ we infer that $|F^{\sigma'}| = |F^\sigma|$, and $F^{\sigma'}$ is also a minimum completion of G . Since F^σ is also lexicographically minimum, it is easy to see that the last criterion of the definition of the value of the layer-one state $(J^\sigma(p^1), J^\sigma(p^2))$ indeed chooses $\sigma|_M$. \square

By Lemma 5.8, our goal is to compute $f[J^\sigma(1), J^\sigma(n-2)]$ by dynamic programming. Observe that both $J^\sigma(1)$ and $J^\sigma(n-2)$ are known, due to the augmentation performed at the beginning of this section.

Our dynamic programming algorithm computes value $g[J^1, J^2]$ for every layer-one state (J^1, J^2) , and we will ensure that $g[J^\sigma(p^1), J^\sigma(p^2)] = f[J^\sigma(p^1), J^\sigma(p^2)]$ for any $p^1 \leq p^2$ with $J^\sigma(p^1), J^\sigma(p^2) \in \mathcal{J}$; we will not necessarily guarantee that the values of f and g are equal for other states. (Formally, $g[J^1, J^2]$ may also take value of \perp ,

which implies that either J^1 or J^2 is not consistent with σ ; we assign this value to $g[J^1, J^2]$ whenever we find no candidate for its value.)

Consider now one layer-one state (J^1, J^2) with $J^1 = (A^1, X^1, \sigma_X^1)$, $J^2 = (A^2, X^2, \sigma_X^2)$. The base case for computing $g[J^1, J^2]$ is the case where $A^2 \subseteq A^1 \cup X^1$. Then $\sigma_X^1 \cup \sigma_X^2$ is the only candidate for the value $f[J^1, J^2]$, provided that σ_X^1 and σ_X^2 agree on the intersection of their domains.

In the other case, we iterate through all possible tuples $J^3 = (A^3, X^3, \sigma_X^3)$, with $A^1 \subset A^3 \subset A^2$ such that both (J^1, J^3) and (J^3, J^2) are layer-one states, and try $g[J^1, J^3] \cup g[J^3, J^2]$ as a candidate value for $g[J^1, J^2]$. That is, we temporarily pick $g[J^1, J^2]$ with the same criteria as for $f[J^1, J^2]$, but taking into account only values $g[J^1, J^3] \cup g[J^3, J^2]$ for different choices of J^3 .

Since the minimization for $g[J^1, J^2]$ is taken over smaller set of functions than for $f[J^1, J^2]$, we infer that

1. the cardinality of $F^{f[J^1, J^2]}$ is not larger than the cardinality of $F^{g[J^1, J^2]}$;
2. even if these two sets are of equal size, $f[J^1, J^2]$ is lexicographically not larger than $g[J^1, J^2]$.

However, observe that if $J^1 = J^\sigma(p^1)$ and $J^2 = J^\sigma(p^2)$ and there exists p^3 such that $p^1 < p^3 < p^2$ and $J^\sigma(p^3) \in \mathcal{J}$, then $g[J^1, J^\sigma(p^3)] \cup g[J^\sigma(p^3), J^2]$ is taken into account when evaluating $g[J^1, J^2]$. If we compute the values for the states (J^1, J^2) in the order of nondecreasing values of $|A^2 \setminus A^1|$, then the values $g[J^1, J^\sigma(p^3)], g[J^\sigma(p^3), J^2]$ have been computed before, and moreover by the induction hypothesis they are equal to $f[J^1, J^\sigma(p^3)]$ and $f[J^\sigma(p^3), J^2]$, respectively. Therefore,

$$f[J^1, J^\sigma(p^3)] \cup f[J^\sigma(p^3), J^2] = \sigma|_{A_{\text{jump}(p^2)} \setminus A_{p^1}}$$

is taken as a candidate value for $g[J^1, J^2]$ and, consequently, $g[J^1, J^2] = f[J^1, J^2] = \sigma|_{A_{\text{jump}(p^2)} \setminus A_{p^1}}$.

Finally, we need to ensure that $g[J^1, J^2] = f[J^1, J^2]$ in the case when such position p^3 does not exist. To this end, we take also more candidate values for $g[J^1, J^2]$, computed by the layer-two dynamic programming in the next section. We ensure that if $J^1 = J^\sigma(p^1)$, $J^2 = J^\sigma(p^2)$ but for any $p^1 < q < p^2$ we have $J^\sigma(q) \notin \mathcal{J}$, then the layer-two dynamic programming actually outputs $f[J^1, J^2]$ as one of the candidates and runs in time $(n|\mathcal{S}|)^{\mathcal{O}(\tau)}$ for any choice of J^1, J^2 . By Theorem 5.6 there are at most $n^{\mathcal{O}(k/\tau)}|\mathcal{S}|^4$ layer-one states. Hence by using $(n|\mathcal{S}|)^{\mathcal{O}(\tau)}$ work for each of them will give the running time promised in Theorem 5.1.

5.2. Layer two: Chains. In this section we are given a layer-one state (J^1, J^2) with $J^1 = (A^1, X^1, \sigma_X^1)$, $J^2 = (A^2, X^2, \sigma_X^2)$; denote $p^\alpha = |A^\alpha| + 1$, $r^\alpha = |A^\alpha \cup X^\alpha| + 1$ for $\alpha = 1, 2$. We are to compute, in time $(n|\mathcal{S}|)^{\mathcal{O}(\tau)}$, the value $f[J^1, J^2]$, assuming $J^1 = J^\sigma(p^1)$, $J^2 = J^\sigma(p^2)$, and for no position $p^1 < q < p^2$ it holds that $J^\sigma(q) \in \mathcal{J}$. By Theorem 5.6, it implies that the number of edges of F^σ incident to any set X_q for $p^1 < q < p^2$ is more than $2k/\tau$. Observe that $X^\alpha = A_{\text{jump}(p^\alpha)} \setminus A_{p^\alpha}$ holds, and hence $r^\alpha = \text{jump}(p^\alpha)$ for $\alpha = 1, 2$.

For any position q , consider the following sequence: $z_q(0) = q$ and $z_q(i + 1) = \text{jump}(z_q(i))$ (with the convention that $\text{jump}(\infty) = \infty$). Observe the following.

LEMMA 5.9. *For any $q \geq p^1$ it holds that $z_q(\tau) \geq p^2$.*

Proof. Consider any $q \geq p^1$. For any $i > 0$ such that $z_q(i) < p^2$ we have that there are more than $2k/\tau$ edges of F^σ incident to $X_{z_q(i)}$. However, the sets $X_{z_q(i)}$ are pairwise disjoint for different values of i . Since $|F^\sigma| \leq k$, we infer that for less than τ values $i > 0$ we may have $z_q(i) < p^2$, and the lemma is proven. \square

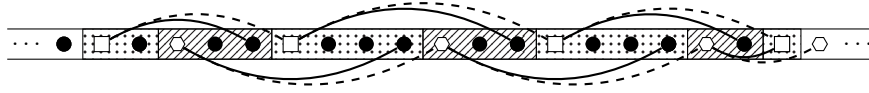


FIG. 8. The separation property provided by Lemma 5.11. The sequences $z_c(i)$ and $z_d(i)$ are denoted with rectangular and hexagonal shapes, respectively. The sets C_i and D_i are denoted with dots and lines, respectively.

By a straightforward induction from Lemma 5.2 we obtain the following.

LEMMA 5.10. For any two positions c, d with $c \leq d \leq \text{jump}(c)$ and for any $i \geq 0$ it holds that

$$z_c(i) \leq z_d(i) \leq z_c(i + 1).$$

The next observation gives us the crucial separation property for the layer-two dynamic programming (see also Figure 8).

LEMMA 5.11. For any positions c, d with $c \leq d \leq \text{jump}(c)$ define

$$C_i = \sigma^{-1}([z_c(i), z_d(i) - 1]),$$

$$D_i = \sigma^{-1}([z_d(i), z_c(i + 1) - 1]).$$

Then

1. sets C_i, D_i form a partition of $V(G) \setminus A_c$;
2. for any $i \geq 0$, it holds that both $C_i \cup D_i$ and $D_i \cup C_{i+1}$ are cliques in G^σ ;
3. for any $j > i \geq 0$ there is no edge in G^σ between C_i and D_j ;
4. for any $i > j + 1 > 0$ there is no edge in G^σ between C_i and D_j .

Proof. All statements follow from the definitions $z_c(i + 1) = \text{jump}(z_c(i))$ and $z_d(i + 1) = \text{jump}(z_d(i))$, and from Lemmata 5.3 and 5.10. \square

Intuitively, Lemma 5.11 implies that we may independently consider the vertices of $\bigcup_{i \geq 0} C_i$ and of $\bigcup_{i \geq 0} D_i$: the sequences $z_c(i)$ and $z_d(i)$ give us some sort of ‘‘horizontal’’ partition of the graphs G and G^σ . We now formalize this idea.

DEFINITION 5.12 (chain). A chain is a quadruple (s, z, u, B) , where

$$s \in \{0, 1, \dots, \tau\},$$

$$z : \{0, 1, \dots, s\} \rightarrow [p^1, r^2],$$

$$u : \{0, 1, \dots, s\} \rightarrow V(G),$$

$$B : \{0, 1, \dots, s\} \rightarrow 2^{V(G)}$$

with the following properties:

1. $z(i) \in [p^2, r^2]$ if and only if $i = s$;
2. $z(i) < z(i + 1)$ for any $0 \leq i < s$;
3. $|B(i)| = z(i) - 1$ for any $0 \leq i \leq s$;
4. $B(i) \subset B(i + 1)$, for any $0 \leq i < s$;
5. $u(i) \in B(j)$ if and only if $0 \leq i < j \leq s$;
6. no edge of G connects a vertex of $B(i)$ with a vertex of $V(G) \setminus B(i + 1)$ for any $0 \leq i < s$.

A chain (s, z, u, B) is consistent with the ordering σ if $s = \min\{i : z_{z(0)}(i) \geq p^2\}$ and for all $0 \leq i \leq s$,

1. $z(i) = z_{z(0)}(i)$;

- 2. $\sigma(u(i)) = z(i)$;
- 3. $B(i) = A_{z(i)}$.

We remark here that if $p^2 \leq n - 2$, then $\text{jump}(q) \leq r^2$ for any $q \leq p^2$, and hence $z_{z(0)}(s) \leq r^2$ for any $z(0) \leq r^2$ in the definition above.

Our next lemma follows immediately from the definition of a chain.

LEMMA 5.13. *For $q \in [p^1, r^2]$, let $s = \min\{i : z_q(i) \geq p^2\}$. For every $0 \leq i \leq s$, let*

$$\begin{aligned} z(i) &= z_q(i), \\ u(i) &= \sigma^{-1}(z(i)), \\ B(i) &= A_{z(i)}. \end{aligned}$$

Then $I^\sigma(q) := (s, z, u, B)$ is a chain consistent with σ .

Moreover, the bound of Lemma 5.9 gives us the following enumeration algorithm.

THEOREM 5.14. *In $(n|\mathcal{S}|)^{\mathcal{O}(\tau)}$ time one can enumerate a family \mathcal{C} of at most $(n|\mathcal{S}|)^{\mathcal{O}(\tau)}$ chains that contains all chains consistent with σ .*

Proof. There are $1 + \tau \leq n$ possible values for s . For each $0 \leq i \leq s$, there are at most n choices for $z(i)$, n choices for $u(i)$, and $|\mathcal{S}|$ choices for $B(i)$. The bound $s \leq \tau$ due to Lemma 5.9 yields the desired bound. Observe that the properties of a chain can be verified in polynomial time. \square

We are now finally ready to state the definition of a layer-two state with its value.

DEFINITION 5.15 (layer-two state). *A layer-two state consists of two chains $I^1 = (s^1, z^1, u^1, B^1)$, $I^2 = (s^2, z^2, u^2, B^2)$ with $I^1, I^2 \in \mathcal{C}$ such that*

- 1. $s^2 \leq s^1 \leq s^2 + 1$,
- 2. $z^1(i) \leq z^2(i)$, $B^1(i) \subseteq B^2(i)$ for any $1 \leq i \leq s^2$ and $z^2(i) \leq z^1(i + 1)$, $B^2(i) \subseteq B^1(i + 1)$ for any $1 \leq i < s^1$;
- 3. $u^1(i) = u^2(j)$ if and only if $z^1(i) = z^2(j)$ for any $1 \leq i \leq s^1$ and $1 \leq j \leq s^2$;

Furthermore, we denote

$$\begin{aligned} C_i[I^1, I^2] &= B^2(i) \setminus B^1(i) && \text{for any } 0 \leq i \leq s^2, \\ D_i[I^1, I^2] &= B^1(i + 1) \setminus B^2(i) && \text{for any } 0 \leq i < s^1, \\ Z_i[I^1, I^2] &= [z^1(i), z^2(i) - 1] && \text{for any } 0 \leq i \leq s^2, \\ C_{s^1}[I^1, I^2] &= (A^2 \cup X^2) \setminus B^1(s^1) && \text{if } s^2 < s^1, \\ Z_{s^1}[I^1, I^2] &= [z^1(s^1), r^2 - 1] && \text{if } s^2 < s^1, \end{aligned}$$

and we require that for any $0 \leq i \leq s^1$ all positions of $Z_i[I^1, I^2]$ are pairwise adjacent in G_\uparrow . We define G_\downarrow^* to be equal to G_\downarrow with additionally $[p^2, r^2 - 1]$ and each $Z_i[I^1, I^2]$ turned into a clique, for every $0 \leq i \leq s^1$. Note that by Lemma 2.3, G_\downarrow^* is a proper interval graph with identity being an umbrella ordering. Moreover, it holds that $E(G_\downarrow^*) \subseteq E(G_\uparrow)$ by the construction of $E(G_\downarrow^*)$ and the fact that $J^2 \in \mathcal{J}$.

The value of a layer-two state (I^1, I^2) is a bijection $f[I^1, I^2] : \bigcup_{i=0}^{s^1} C_i[I^1, I^2] \rightarrow \bigcup_{i=0}^{s^1} Z_i[I^1, I^2]$ such that

- 1. $f[I^1, I^2]$ is a feasible ordering of its domain, that is, for any u in the domain of $f[I^1, I^2]$ we have $f[I^1, I^2](u) \in \Sigma_u$, and for any u_1, u_2 in the domain of $f[I^1, I^2]$ such that $u_1 u_2 \in E(G)$, it holds that $f[I^1, I^2](u_1) f[I^1, I^2](u_2) \in E(G_\uparrow)$;
- 2. $f[I^1, I^2](u) \in Z_i[I^1, I^2]$ whenever $u \in C_i[I^1, I^2]$;
- 3. $f[I^1, I^2](u^1(i)) = z^1(i)$ for all $0 \leq i \leq s^1$;

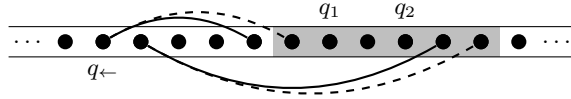


FIG. 9. The last case of the definition of a relevant pair (q_1, q_2) . Both positions q_1 and q_2 need to belong to the gray area.

4. $f[I^1, I^2](u) = \sigma_X^1(u)$ whenever $u \in X^1$ and $f[I^1, I^2](u) = \sigma_X^2(u)$ whenever $u \in X^2$;
5. among all functions f satisfying the previous conditions, $f[I^1, I^2]$ minimizes the cardinality of $F^{f,*}$, where the set $F^{f,*}$ is defined as the unique minimal completion for the ordering f of the subgraph of G induced by the domain of f and SPIC instance $(G, k, (\Sigma_u)_{u \in V(G)}, G_{\downarrow}^*, G_{\uparrow})$;
6. among all functions f satisfying the previous conditions, $f[I^1, I^2]$ is lexicographically minimum.

Note that in the definition of a layer-two state we *do not* require that any of the chains begins in $[p^1, r^1]$, i.e., that $z^1(0)$ or $z^2(0)$ are in this interval. The values for the states where these chains begin at arbitrary positions within $[p^1, r^2]$ will be essential for computing the final value we are interested in.

Similarly as in the case of layer-one states, we have the following claim.

DEFINITION 5.16 (relevant pair). A pair (q^1, q^2) with $p^1 \leq q^1 \leq q^2 \leq \min(\text{jump}(q^1), r^2)$ is called relevant if one of the following holds:

1. $q^2 \leq r^1$,
2. $q^1 = q^2$, or
3. there exists a position $q_{\leftarrow} \geq p^1$ such that $\text{jump}(q_{\leftarrow}) \leq q^1 \leq q^2 \leq \text{jump}(q_{\leftarrow} + 1)$ (see also Figure 9).

LEMMA 5.17. For any q_1, q_2 such that $p^1 \leq q^1 \leq q^2 \leq \min(\text{jump}(q^1), r^2)$, the pair $(I^\sigma(q^1), I^\sigma(q^2))$ is a layer-two state. If moreover (q^1, q^2) is a relevant pair, then $f[I^\sigma(q^1), I^\sigma(q^2)]$ is a restriction of σ to the domain of $f[I^\sigma(q^1), I^\sigma(q^2)]$. In particular, $f[I^\sigma(p^1), I^\sigma(r^1)] = f[J^\sigma(p^1), J^\sigma(p^2)]$.

Proof. By somehow abusing the notation, we denote $X^1 = X_{p^1}$ and $X^2 = X_{p^2}$. It is straightforward to verify from the definition that $(I^\sigma(q^1), I^\sigma(q^2))$ is a layer-two state and the restriction of σ to $Y := \bigcup_i C_i[I^\sigma(q^1), I^\sigma(q^2)]$ satisfies the first four requirements of the definition of a value of a layer-two state, even if (q^1, q^2) is not a relevant pair. Moreover, observe that Lemma 5.11 implies that $F^\sigma \cap \binom{Y}{2}$ is a completion for the ordering $\sigma|_Y$ of Y in the instance $(G, k, (\Sigma_u)_{u \in V(G)}, G_{\downarrow}^*, G_{\uparrow})$. Hence, $F^{\sigma|_Y,*} \subseteq F^\sigma \cap \binom{Y}{2}$.

Now assume that (q^1, q^2) is a relevant pair and denote $f = f[I^\sigma(q^1), I^\sigma(q^2)]$ and $I^\sigma(q^\alpha) = (s^\alpha, z^\alpha, u^\alpha, B^\alpha)$ for $\alpha = 1, 2$. If $q^1 = q^2$, then observe that the sets $C_i[I^\sigma(q^1), I^\sigma(q^2)]$ are empty, and the state in question asks for an empty function. Hence, assume $q^1 < q^2$. Define an ordering σ' of $V(G)$ so that $\sigma'(u) = f(u)$ for any $u \in Y$ and $\sigma'(u) = \sigma(u)$ otherwise.

Let us define $F := (F^\sigma \setminus \binom{Y}{2}) \cup F^{f,*}$. In the subsequent claims we establish some properties of the graph $G + F$ and ordering σ' .

CLAIM 1. $\sigma'(G + F)[[1, r^1 - 1] \cup [p^2, n]] = \sigma(G^\sigma)[[1, r^1 - 1] \cup [p^2, n]]$.

Proof. Note here that σ' and σ agree on positions before r^1 and after $p^2 - 1$. Observe also that $[p^1, r^1 - 1]$ and $[p^2, r^2 - 1]$ are cliques in $\sigma(G^\sigma)$, and $[p^1, r^1 - 1]$ can have a nonempty intersection only with the first of the intervals $Z_i[I^\sigma(q^1), I^\sigma(q^2)]$.

Since $([p^2, r^2 - 1], (Z_i[I^\sigma(q^1), I^\sigma(q^2)])) \subseteq E(G_\downarrow^*) \cap (\sigma(Y)) \subseteq E(\sigma'(G[Y] + F^{f,*}))$, it follows by the definition of F that that intervals $[p^1, r^1 - 1]$ and $[p^2, r^2 - 1]$ are cliques in $\sigma'(G + F)$ as well. Since $Y \subseteq \sigma^{-1}([p^1, r^2 - 1])$, the claim follows. \square

CLAIM 2. σ' is a umbrella ordering of $G + F$.

Proof. Consider any $a, b, c \in V(G)$ with $ac \in E(G + F)$ and $\sigma'(a) < \sigma'(b) < \sigma'(c)$; we want to show the umbrella property for the triple a, b, c in the graph $G + F$. We consider a few cases, depending on the intersection $\{a, b, c\} \cap Y$.

1. If $a, b, c \in Y$ or $a, b, c \notin Y$, then the umbrella property holds by the definition of F^σ and $F^{f,*}$.
2. If $\sigma'(a) \geq p^2$ or $\sigma'(c) < r^1$, then recall that $\sigma'(G + F)[[1, r^1 - 1] \cup [p^2, n]] = \sigma(G^\sigma)[[1, r^1 - 1] \cup [p^2, n]]$. Then the umbrella property for a, b, c follows from the fact that σ is an umbrella ordering of G^σ .

Hence, in the remaining cases we have in particular that $a \notin X^2$ and $c \notin X^1$. Observe also that the assumption $ac \in E(G + F)$ implies that $p^1 \leq \sigma'(a) < \sigma'(c) < r^2$, since $r^1 = \text{jump}(p^1)$ and $r^2 = \text{jump}(p^2)$.

3. If $a, c \in Y$ and $b \notin Y$, then, by the structure of Y , we have $a \in C_i[I^\sigma(q^1), I^\sigma(q^2)]$, $c \in C_j[I^\sigma(q^1), I^\sigma(q^2)]$ for some $1 \leq i < j \leq s^1$. We claim that $j = i + 1$. Assume the contrary. Observe that if $i + 1 < j$, then in particular $i < s^2$. By Lemma 5.11, no edge of G^σ connects $A_{z_{q^2}(i)}$ with $V(G) \setminus A_{z_{q^2}(i+1)}$, so in particular there is no such edge either in G , which is a subgraph of G^σ . Likewise, there is no edge between $[1, z_{q^2}(i)]$ and $[z_{q^2}(i + 1), n]$ in G_\downarrow^* . By the construction of $F^{f,*}$ it follows that also no edge of $F^{f,*}$ connects $A_{z_{q^2}(i)}$ with $V(G) \setminus A_{z_{q^2}(i+1)}$. As σ and σ' differ only on the internal ordering of each set $C_i[I^\sigma(q^1), I^\sigma(q^2)]$, and $ac \in E(G + F)$, we have a contradiction, and hence $c \in C_{i+1}[I^\sigma(q^1), I^\sigma(q^2)]$. It follows that $b \in D_i[I^\sigma(q^1), I^\sigma(q^2)]$ and, by Lemma 5.11, $ab, bc \in E(G^\sigma)$. By the definition of F , $ab, bc \in E(G) \cup F$.

In the remaining cases we have that either a or c does not belong to Y . Hence $ac \in E(G^\sigma)$ by the definition of F .

4. If $a \in Y \setminus X^2$ and $c \notin Y$, then, by Lemma 5.11, we have $a \in C_i[I^\sigma(q^1), I^\sigma(q^2)]$ and $c \in D_i[I^\sigma(q^1), I^\sigma(q^2)]$ for some $0 \leq i < s^1$. By Lemma 5.11, $C_i[I^\sigma(q^1), I^\sigma(q^2)] \cup D_i[I^\sigma(q^1), I^\sigma(q^2)]$ is a clique in G^σ , and, by the definition of G_\downarrow^* , $C_i[I^\sigma(q^1), I^\sigma(q^2)]$ is a clique in $G + F$. Hence, $ab, bc \in E(G) \cup F$ regardless of whether $b \in Y$.
5. If $a \notin Y$ and $c \in C_i[I^\sigma(q^1), I^\sigma(q^2)]$ for some $i > 0$, then, by Lemma 5.11, $a \in D_{i-1}[I^\sigma(q^1), I^\sigma(q^2)]$. As in the previous case, Lemma 5.11 asserts that $D_{i-1}[I^\sigma(q^1), I^\sigma(q^2)] \cup C_i[I^\sigma(q^1), I^\sigma(q^2)]$ is a clique in G^σ , and the definition of G_\downarrow^* gives us that $C_i[I^\sigma(q^1), I^\sigma(q^2)]$ is a clique in $G + F$. Consequently, $ab, bc \in E(G) \cup F$ regardless of whether $b \in Y$.
6. If $a \notin Y$ and $c \in C_0[I^\sigma(q^1), I^\sigma(q^2)] = \sigma^{-1}([q^1, q^2 - 1])$, then, as $\sigma'(c) \geq r^1$, we have that pair (q^1, q^2) is a relevant pair due to the existence of some position q_{\leftarrow} . Since $ac \in E(G^\sigma)$, we have that $\sigma'(a) = \sigma(a) \geq q_{\leftarrow} + 1$. As $\text{jump}(q_{\leftarrow} + 1) \geq q^2$, we have that also $ab \in E(G^\sigma)$ and $bc \in E(G^\sigma)$. By the definition of F we infer that $ab \in E(G) \cup F$ and, additionally, $bc \in E(G) \cup F$ in the case $b \notin Y$. If $b \in Y$, then $b \in C_0[I^\sigma(q^1), I^\sigma(q^2)]$ and $bc \in E(G) \cup F$ by the definition of G_\downarrow^* .
7. If $a, c \notin Y$ and $b \in Y$, then let $b \in C_i[I^\sigma(q^1), I^\sigma(q^2)]$ for some $0 \leq i \leq s^1$. Since σ and σ' differ only on internal ordering of sets $C_i[I^\sigma(q^1), I^\sigma(q^2)]$ and $a, c \notin Y$, then the condition $\sigma'(a) < \sigma'(b) < \sigma'(c)$ implies also $\sigma(a) < \sigma(b) < \sigma(c)$. Since $ac \in E(G^\sigma)$ and σ is an umbrella ordering of G^σ , we infer that

$ab, bc \in E(G^\sigma)$. By the definition of F this implies that $ab, bc \in E(G + F)$. \square

CLAIM 3. $E(G_\downarrow^*) \subseteq E(\sigma'(G + F))$.

Proof. Consider any $pq \in E(G_\downarrow^*)$. Denote $a = \sigma^{-1}(p)$, $b = \sigma^{-1}(q)$ and similarly denote a' and b' for the ordering σ' ; we want to show that $a'b' \in E(G) \cup F$. As $E(G_\downarrow^*) \subseteq E(\sigma(G^\sigma))$ we have $ab \in E(G^\sigma)$. If $p, q \in \bigcup_i Z_i[I^\sigma(q^1), I^\sigma(q^2)]$, then $a'b' \in E(G) \cup F^{f,*}$ by the definition of $F^{f,*}$. Otherwise, without loss of generality assume that $q \notin \bigcup_i Z_i[I^\sigma(q^1), I^\sigma(q^2)]$, and hence $b = b'$. If additionally $a = a'$, then $a'b' \in E(G) \cup F$ follows directly from the definition of F and the fact that $ab \in E(G^\sigma)$. In the remaining case, if $a \neq a'$, we have $p \in Z_i[I^\sigma(q^1), I^\sigma(q^2)]$ and $a, a' \in C_i[I^\sigma(q^1), I^\sigma(q^2)]$ for some $0 \leq i \leq s^1$. Moreover, from the assumption $a \neq a'$ we infer that $r^1 \leq p < p^2$, and consequently $i < s^1$. By the definition of F , we need to show that $a'b \in E(G^\sigma)$.

We consider two cases, depending on the relative order of p and q . If $p < q$, then we have $z^2(i) \leq q < z^1(i+1)$ by Lemma 5.11 and consequently $b \in D_i[I^\sigma(q^1), I^\sigma(q^2)]$. By Lemma 5.11 again, b is adjacent to all vertices of $C_i[I^\sigma(q^1), I^\sigma(q^2)]$ in the graph G^σ , and $a'b \in E(G^\sigma)$. A similar argument holds if $q < p$ and $i > 0$: by Lemma 5.11, we have first that $b \in D_{i-1}[I^\sigma(q^1), I^\sigma(q^2)]$ and second that b is adjacent in G^σ to all vertices of $C_i[I^\sigma(q^1), I^\sigma(q^2)]$, and hence $a'b \in E(G^\sigma)$. In the remaining case, if $q < p$ and $i = 0$ (hence $p \in [q^1, q^2 - 1]$), from $p \geq r^1$ it follows that the reason (q^1, q^2) is a relevant pair is existence of some position q_\leftarrow . Since $ab \in E(G^\sigma)$, we infer that $q \geq q_\leftarrow + 1$. Hence, b is adjacent in G^σ to all vertices of $C_0[I^\sigma(q^1), I^\sigma(q^2)]$, in particular to a' , and the claim is proven. \square

CLAIM 4. $E(\sigma'(G)) \subseteq G_\uparrow$ and σ' is a feasible ordering of G .

Proof. Observe that it follows directly from the definition of σ' that $\sigma'(u) \in \Sigma_u$ for any vertex u . Hence, to show feasibility of σ' it suffices to show that $E(\sigma'(G)) \subseteq G_\uparrow$.

Consider any $ab \in E(G)$. If both a and b belong to Y or both do not belong, then the claim is obvious by the feasibility of both σ and f . Assume then $a \in Y$ and $b \notin Y$. If $\sigma(a) = \sigma'(a)$, then clearly $\sigma'(a)\sigma'(b) = \sigma(a)\sigma(b) \in E(G_\uparrow)$. Otherwise, $a \notin X^2$ and $a \in C_i[I^\sigma(q^1), I^\sigma(q^2)]$ for some $0 \leq i < s^1$. If $\sigma(b) \geq z_{q^2}(i)$, then Lemma 5.11 implies that $b \in D_i[I^\sigma(q^1), I^\sigma(q^2)]$. By Lemma 5.11 again, $[z_{q^1}(i), z_{q^1}(i+1) - 1]$ is a clique in $\sigma(G^\sigma)$ and hence in G_\uparrow as well, so $\sigma'(a)\sigma'(b) \in E(G_\uparrow)$. A similar situation happens if $\sigma(b) < z_{q^1}(i)$ and $i > 0$: $b \in D_{i-1}[I^\sigma(q^1), I^\sigma(q^2)]$ and again Lemma 5.11 together with feasibility of σ proves the claim. In the remaining case $i = 0$ and $\sigma(b) < q^1$. As $\sigma(a) \neq \sigma'(a)$ we have $a \notin X^1$ and hence the reason (q^1, q^2) is a relevant pair must be existence of some position q_\leftarrow . As $ab \in E(G)$ we have $\sigma(b) \geq q_\leftarrow + 1$. As $\text{jump}(q_\leftarrow + 1) \geq q^2$, the position b is adjacent to all positions of $[q^1, q^2 - 1]$ in $\sigma(G^\sigma)$ and hence $\sigma'(a)\sigma'(b) \in E(\sigma(G^\sigma)) \subseteq E(G_\uparrow)$ as claimed. \square

From the above claims we infer that $|F^{\sigma'}| \leq |F^{f,*}| + |F^\sigma \setminus \binom{Y}{2}|$, whereas $F^{\sigma|_Y,*} \subseteq F^\sigma \cap \binom{Y}{2}$. By the minimality of both f and σ , including the lexicographical minimality, we have $f = \sigma|_Y$ and the lemma is proven. \square

The layer-two dynamic programming algorithm computes, for any layer-two state (I^1, I^2) , a function $g[I^1, I^2]$ that satisfies the first four conditions of $f[I^1, I^2]$, and we will inductively ensure that $g[I^\sigma(q^1), I^\sigma(q^2)] = f[I^\sigma(q^1), I^\sigma(q^2)]$ for any relevant pair (q^1, q^2) . We compute the values $g[I^1, I^2]$ in the order of decreasing value of $z^1(0)$ and, subject to that, increasing value of $z^2(0)$. (Formally, $g[I^1, I^2]$ may also take value of \perp , which implies that either I^1 or I^2 is not consistent with σ ; we assign this value to $g[I^1, I^2]$ whenever we find no candidate for its value.)

Consider now a fixed layer-two state (I^1, I^2) with $I^1 = (s^1, z^1, u^1, B^1)$ and $I^2 = (s^2, z^2, u^2, B^2)$. We start with the the base case when we have that either $z^1(0) =$

$z^2(0)$ or $z^1(0) \geq p^2 - 1$. Observe that in this situation we have that the domain of $g[I^1, I^2]$ is either X^2 or X^2 with an additional element $u^1(0)$ which must be mapped to $z^1(0) = p_2 - 1$. Hence all the values of $f[I^1, I^2]$ are fixed by σ_X^2 , u^1 , and z^1 , and there is only one candidate for this value. It is straightforward to verify that, in the case when $I^1 = I^\sigma(q^1)$ and $I^2 = I^\sigma(q^2)$, this unique candidate is indeed a restriction of σ and hence equals $f[I^1, I^2]$.

In the inductive step we have $z^1(0) < z^2(0)$ and $z^1(0) < p^2 - 1$. We consider two cases, depending on the value of $z^2(0) - z^1(0)$.

First assume $z^2(0) - z^1(0) > 1$. In this case consider all possible chains $I^3 = (s^3, z^3, u^3, B^3)$ such that both (I^1, I^3) and (I^3, I^2) are layer-two states, and $z^1(0) < z^3(0) < z^2(0)$. We take as a candidate value for $g[I^1, I^2]$ the union $g[I^1, I^3] \cup g[I^3, I^2]$ and pick $g[I^1, I^2]$ using the criteria from the definition of the value $f[I^1, I^2]$, but taking only functions $g[I^1, I^3] \cup g[I^3, I^2]$ for all choices of I^3 as candidates.

We claim that if $I^1 = I^\sigma(q^1)$, $I^2 = I^\sigma(q^2)$ and (q^1, q^2) is a relevant pair, then $g[I^1, I^2] = f[I^1, I^2]$. Note that it suffices to show that $f[I^1, I^2]$ is considered as a candidate for $g[I^1, I^2]$ in the aforementioned process for some choice of I^3 . Consider any $q^1 < q^3 < q^2$ and observe that if (q^1, q^2) is a relevant pair, then also (q^1, q^3) and (q^3, q^2) are relevant pairs: this is clearly true for the case $q^2 \leq r^1$ and, in the last case of the definition of a relevant pair, notice that the same position q_{\leftarrow} witnesses also that (q^1, q^3) and (q^3, q^2) are relevant. Denote $I^3 = I^\sigma(q^3)$ and observe that we consider a candidate $g[I^1, I^3] \cup g[I^3, I^2]$ for $g[I^1, I^2]$. By Lemma 5.17 and the inductive assumption, this candidate is a restriction of σ and hence, again by Lemma 5.17, equals $f[I^1, I^2]$.

We are left with the case $z^2(0) = z^1(0) + 1$. As $z^1(0) < p^2 - 1$, we have $z^2(0) < p^2$. For $\alpha = 1, 2$ define $s_*^\alpha = s^\alpha - 1$, and $z_*^\alpha(i) = z^\alpha(i + 1)$, $u_*^\alpha(i) = u^\alpha(i + 1)$ and $B_*^\alpha(i) = B^\alpha(i + 1)$ for any $0 \leq i \leq s_*^\alpha$, and $I_*^\alpha = (s_*^\alpha, z_*^\alpha, u_*^\alpha, B_*^\alpha)$. In this case we consider only one candidate for $g[I^1, I^2]$, being $g[I_*^1, I_*^2]$, extended with $g[I^1, I^2](u^1(0)) = z^1(0)$.

It remains to show that if $I^1 = I^\sigma(q^1)$, $I^2 = I^\sigma(q^2)$ and (q^1, q^2) is an relevant pair, then $g[I^1, I^2] = f[I^1, I^2]$. Observe that $I_*^1 = I^\sigma(\text{jump}(q^1))$ and $I_*^2 = I^\sigma(\text{jump}(q^2))$. Moreover, the position q^1 witnesses that $(\text{jump}(q^1), \text{jump}(q^2))$ is a relevant pair, and hence $g[I_*^1, I_*^2] = f[I_*^1, I_*^2]$ by induction. This completes the proof that $g[I^\sigma(q^1), I^\sigma(q^2)] = f[I^\sigma(q^1), I^\sigma(q^2)]$ for all relevant pairs (q^1, q^2) .

As candidates for the value $f[J^1, J^2]$ of the layer-one state (J^1, J^2) we are currently processing, we take all the values $g[I^1, I^2]$ for all the layer-two states (I^1, I^2) for which the domain of $f[I^1, I^2]$ is equal to the domain of $f[J^1, J^2]$. By Theorem 5.14, there are at most $(n|\mathcal{S}|)^{\mathcal{O}(\tau)}$ guesses for such states, and they can be enumerated in $(n|\mathcal{S}|)^{\mathcal{O}(\tau)}$ time. Observe also that if indeed $J^1 = J^\sigma(p^1)$ and $J^2 = J^\sigma(p^2)$, then the layer-two state $(I^1, I^2) = (I^\sigma(p^1), I^\sigma(r^1))$ will be among the enumerated states. Since (p^1, r^1) is a relevant pair, we have that $g[I^\sigma(p^1), I^\sigma(r^1)] = f[I^\sigma(p^1), I^\sigma(r^1)]$, while by Lemma 5.17 we have that $f[I^\sigma(p^1), I^\sigma(r^1)]$ is equal to the restriction of σ to its domain, which in turn is equal to the domain of $g[J^1, J^2]$. Hence, the restriction of σ to the domain of $g[J^1, J^2]$, which is exactly equal to $f[J^1, J^2]$ by Lemma 5.8, will be among the enumerated candidate values—this was exactly the property needed by the layer-one dynamic program.

By Theorem 5.14 there are $(n|\mathcal{S}|)^{\mathcal{O}(\tau)}$ layer-two states; thus the entire computation of $f[J^1, J^2]$ takes $(n|\mathcal{S}|)^{\mathcal{O}(\tau)}$ time, as was promised. This concludes the proof of Theorem 5.1 and hence finishes the proof of Theorem 1.1.

6. Conclusions. We have presented the first subexponential algorithm for PROPER INTERVAL COMPLETION, running in time $k^{\mathcal{O}(k^{2/3})} + \mathcal{O}(nm(kn + m))$. As

many algorithms for completion problems in similar graph classes [3, 7, 9, 10] run in time $\mathcal{O}^*(k^{\mathcal{O}(\sqrt{k})})$, it is tempting to ask for such a running time also in our case. The bottleneck in our approach is the trade-offs between the two layers of dynamic programming.

Also, observe that every $\mathcal{O}^*(2^{\mathcal{O}(\sqrt{k})})$ -time algorithm for PIC would be in fact also a $2^{\mathcal{O}(n)}$ -time algorithm. Since existence of such an algorithm seems unlikely, we would like to ask for a $2^{\Omega(\sqrt{k})}$ lower bound, under the assumption of the Exponential Time Hypothesis. Note that no such lower bound is known for any other completion problem for related graph classes.

REFERENCES

- [1] N. ALON, D. LOKSHTANOV, AND S. SAURABH, *Fast FAST*, in Proceedings of the 36th Colloquium of Automata, Languages and Programming (ICALP), Lecture Notes in Comput. Sci. 5555, Springer, New York, 2009, pp. 49–58.
- [2] S. BESSY AND A. PEREZ, *Polynomial kernels for proper interval completion and related problems*, Inform. Comput., 231 (2013), pp. 89–108.
- [3] I. BLIZNETS, F. V. FOMIN, M. PILIPCZUK, AND M. PILIPCZUK, *A Subexponential Parameterized Algorithm for Interval Completion*, CoRR, abs/1402.3473. 2014.
- [4] I. BLIZNETS, F. V. FOMIN, M. PILIPCZUK, AND M. PILIPCZUK, *A subexponential parameterized algorithm for proper interval completion*, in Algorithms, ESA 2014, 22th Annual European Symposium, Wrocław, Poland, A. S. Schulz and D. Wagner, eds., Lecture Notes in Comput. Sci. 8737, Springer, New York, 2014, pp. 173–184.
- [5] L. CAI, *Fixed-parameter tractability of graph modification problems for hereditary properties*, Inform. Process. Lett., 58 (1996), pp. 171–176.
- [6] E. D. DEMAINÉ, F. V. FOMIN, M. HAJIAGHAYI, AND D. M. THILIKOS, *Subexponential parameterized algorithms on graphs of bounded genus and H -minor-free graphs*, J. Assoc. Comput. Mach., 52 (2005), pp. 866–893.
- [7] P. G. DRANGE, F. V. FOMIN, M. PILIPCZUK, AND Y. VILLANGER, *Exploring subexponential parameterized complexity of completion problems*, in 31st International Symposium on Theoretical Aspects of Computer Science (STACS 2014), Leibniz International Proceedings in Informatics 25 Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, Dagstuhl, Germany, 2014, pp. 288–299.
- [8] U. FEIGE, *Coping with the NP-hardness of the graph bandwidth problem*, in Proceedings of SWAT, 2000, pp. 10–19.
- [9] F. V. FOMIN AND Y. VILLANGER, *Subexponential parameterized algorithm for minimum fill-in*, SIAM J. Comput., 42 (2013), pp. 2197–2216.
- [10] E. GHOSH, S. KOLAY, M. KUMAR, P. MISRA, F. PANOLAN, A. RAI, AND M. S. RAMANUJAN, *Faster parameterized algorithms for deletion to split graphs*, Algorithmica, 71 (2015), pp. 989–1006.
- [11] P. W. GOLDBERG, M. C. GOLUMBIC, H. KAPLAN, AND R. SHAMIR, *Four strikes against physical mapping of DNA*, J. Comput. Biol., 2 (1995), pp. 139–152.
- [12] M. C. GOLUMBIC, *Algorithmic Graph Theory and Perfect Graphs*, Academic Press, New York, 1980.
- [13] R. IMPAGLIAZZO, R. PATURI, AND F. ZANE, *Which problems have strongly exponential complexity?*, J. Comput. System Sci., 63 (2001), pp. 512–530.
- [14] H. KAPLAN, R. SHAMIR, AND R. E. TARJAN, *Tractability of parameterized completion problems on chordal, strongly chordal, and proper interval graphs*, SIAM J. Comput., 28 (1999), pp. 1906–1922.
- [15] S. KRATSCHE AND M. WAHLSTRÖM, *Two edge modification problems without polynomial kernels*, Discrete Optim., 10 (2013), pp. 193–199.
- [16] Y. LIU, J. WANG, C. XU, J. GUO, AND J. CHEN, *An effective branching strategy based on structural relationship among multiple forbidden induced subgraphs*, J. Combin. Optim., 29 (2015), pp. 257–275.
- [17] P. J. LOOGES AND S. OLARIU, *Optimal greedy algorithms for indifference graphs*, Comput. Math. Appl., 25 (1993), pp. 15–25.

- [18] F. S. ROBERTS, *Indifference Graphs*, in Proof Techniques in Graph Theory: Proceedings of the Second Ann Arbor Graph Theory Conference, Academic Press, New York, 1969, pp. 139–146.
- [19] Y. VILLANGER, P. HEGGERNES, C. PAUL, AND J. A. TELLE, *Interval completion is fixed parameter tractable*, SIAM J. Comput., 38 (2009), pp. 2007–2020.
- [20] M. YANNAKAKIS, *Computing the minimum fill-in is NP-complete*, SIAM J. Algebraic Discrete Methods, 2 (1981), pp. 77–79.