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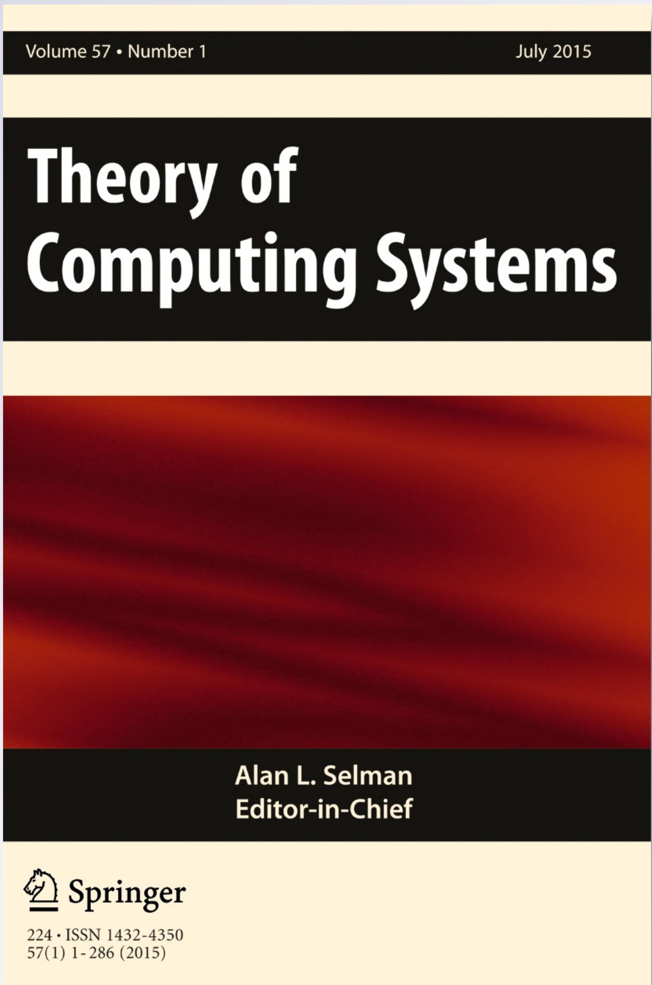

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Minimizing Rosenthal Potential in Multicast Games

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Abstract A multicast game is a network design game modelling how selfish non-cooperative agents build and maintain one-to-many network communication. There is a special source node and a collection of agents located at corresponding terminals. Each agent is interested in selecting a route from the special source to its terminal minimizing the cost. The mutual influence of the agents is determined by a cost sharing mechanism, which evenly splits the cost of an edge among all the agents using it for routing. In this paper we provide several algorithmic and complexity results on finding a Nash equilibrium minimizing the value of Rosenthal potential. Let n be the number of agents and G be the communication network. We show that for a given strategy profile s and integer $k \geq 0$, there is a local search algorithm which in time $n^{O(k)} \cdot |G|^{O(1)}$ finds a better strategy profile, if there is any, in a k-exchange neighbourhood of s. In other words, the algorithm decides if Rosenthal potential can be decreased by changing strategies of at most k agents. The running time of our local search algorithm is essentially tight: unless $FPT = W[1]$, for any function $f(k)$, searching of the k-neighbourhood cannot be done in time $f(k) \cdot |G|^{O(1)}$. We also show that an equilibrium with minimum potential can be found in $3^n \cdot |G|^{O(1)}$ time.

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1 Introduction

Modern networks are often designed and used by non-cooperative individuals with diverse objectives. A considerable part of Algorithmic Game Theory focuses on optimization in such networks with selfish users [2, 6, 9, 13, 14, 17, 22, 23].

In this paper we study the conceptually simple but mathematically rich cost-sharing model introduced by Anshelevich et al. [3, 4], see also [16, Chapter 12]. In a variant of the cost-sharing game, which was called by Chekuri et al. the *multicast game* [5], the network is represented by a weighted directed graph with a distinguished source node r, and a collection of n agents located at corresponding terminals. Each agent is trying to select a cheapest route from r to its terminal. The mutual influence of the players is determined by a cost sharing mechanism identifying how the cost of each edge in the network is shared among the agents using this edge. When h agents use an edge e of cost c_e, each of them has to pay c_e/h. This is a very natural cost sharing formula which is also the outcome of the Shapley value.

The multicast game belongs to the widely studied class of congestion games. This class of games was defined by Rosenthal [21], who also proved that every congestion game has a Nash equilibrium. Rosenthal showed that for every congestion game it is possible to define a potential function which decreases if a player improves its selfish cost. Best-response dynamics in these games always lead to a set of paths that forms a Nash equilibrium. Furthermore, every local minimum of Rosenthal potential corresponds to a Nash equilibrium and vice versa. However, while we know that the multicast game always has a Nash equilibrium, the number of iterations in best-response dynamics achieving an equilibrium can be exponential (see [3, Theorem 5.1]), and it is an important open question if any Nash equilibrium can be found in polynomial time. The next step in the study of Rosenthal potential was done by Anshelevich et al. [3], who showed that Rosenthal potential can be used not only for proving the existence of a Nash equilibrium but also to estimate the quality of equilibrium. Anshelevich et al. defined the price of stability, as the ratio of the best Nash equilibrium cost and the optimum network cost, the *social optimum*. In particular, the cost of a Nash equilibrium minimizing Rosenthal potential is within $\log n$-factor of the social optimum, and thus the global minimum of the potential brings to a cheap equilibrium. The computational complexity of finding a Nash equilibrium achieving the bound of $\log n$ relative to the social optimum is still open, while computing the global minimum of the Rosenthal potential is NP-hard [3, 5].

Our results. In this paper we analyze the following local search problem. Given a strategy profile s, we ask whether a profile with a smaller value of Rosenthal potential can be found in a k-exchange neighbourhood of s, which is the set of all profiles that can be obtained from s by changing strategies of at most k players. Our motivation to study this problem is two-fold.

- If we succeed in finding some Nash equilibrium, say by implementing best-response dynamics, which is still far from the social optimum, it is an important question if the already found equilibrium can be used to find a better one efficiently. Local search heuristic in this case is a natural approach.

- Since the number of iterations in best-response dynamics scenario can be exponential (see [3, Theorem 5.1]), it can be useful to combine the best-response dynamics with a heuristic that at some moments tries to make "larger jumps", i.e., instead of decreasing Rosenthal potential by changing strategy of one player, to decrease the potential by changing in one step strategies of several players.

Let us remark that the number of paths, and thus strategies, every player can select from, is exponential, so the size of the search space also can be exponential. Since the size of k-exchange neighbourhood is exponential, it is not clear a priori, if searching of a smaller value of Rosenthal potential in a k-exchange neighbourhood of a given strategy profile can be done in polynomial time. We show that for a fixed k, the local search can be performed in polynomial time. The running time of our algorithm is $n^{O(k)} \cdot |G|^{O(1)}$, where n is the total number of players[1]. As a subroutine, our algorithm uses a fixed-parameter algorithm computing in time $3^k \cdot |G|^{O(1)}$ the minimum value of Rosenthal potential that can be achieved by changing strategies of at most k agents if a set of k agents whose strategies could be modified is given. We find this auxiliary algorithm to be interesting in its own because it implies that an equilibrium with minimum potential can be found in $3^n \cdot |G|^{O(1)}$ time. It is known that for a number of local search algorithms, exploration of the k-exchange neighbourhood can be done by fixed-parameter tractable (in k) algorithms [10, 18, 24]. We show that, unfortunately, this is not the case for the local search of Rosenthal potential minimum. We use tools from Parameterized Complexity, to show that the running time of our local search algorithm is essentially tight: unless $FPT = W[1]$, searching of the k-neighbourhood cannot be done in time $f(k) \cdot |G|^{O(1)}$ for any function $f(k)$.

2 Preliminaries

Graphs. We consider finite directed and undirected graphs without loops or multiple edges. The vertex set of a (directed) graph G is denoted by $V(G)$, the edge set of an undirected graph and the arc set of a directed graph G is denoted by $E(G)$. To distinguish edges and arcs, the edge with two end-vertices u, v is denoted by uv, and we write (u, v) for the corresponding arc. Let G be a directed graph. For a vertex $v \in V(G)$, we say that u is an *in-neighbor* of v if $(u, v) \in E(G)$. The set of all in-neighbors of v is denoted by $N_G^-(v)$. The *in-degree* $d_G^-(v) = |N_G^-(v)|$. Respectively, u is an *out-neighbor* of v if $(v, u) \in E(G)$, the set of all out-neighbors of v is denoted by $N_G^+(v)$, and the *out-degree* $d_G^+(v) = |N_G^+(v)|$. For a directed graph G, a (directed) *walk* is sequence $v_0, e_1, v_1, e_2, \ldots, e_k, v_k$ of vertices and arcs of G such that $v_0, \ldots, v_k \in V(G)$, $e_1, \ldots, e_k \in E(G)$, and for $i \in \{1, \ldots, k\}$, $e_i = (v_{i-1}, v_i)$. A walk is a (directed) path if all its edges and all its vertices (with possible exception that $v_0 = v_k$) are pairwise distinct. The vertices v_0 and v_k are called *end-vertices*. We say that a walk (path) with end-vertices u and v is a (u, v)-walk (path). We say that a subdigraph T of G is an *out-tree* if T is a directed tree with only one vertex

[1]The number of arithmetic operations used by our algorithms does not depend on the size of the input weights, i.e. the claimed running times are in the unit-cost model.

r of in-degree zero (called the *root*). The vertices of T of out-degree zero are called *leaves*.

Multicast game and Rosenthal potential. A network is modeled by a directed $G = (V, E)$ graph. There is a special *root* or *source* node $r \in V$. There are n multicast users, *players*, and each player has a specified *terminal* node t_i (several players can have the same terminals). A strategy s^i for player i is a path P_i from r to t_i in G. We denote by Π the set of players and by S^i the finite set of strategies of player i, which is the set of all paths from r to t_i. The joint strategy space $S = S^1 \times S^2 \times \cdots \times S^n$ is the Cartesian product of all the possible strategy profiles. At any given moment, a strategy profile (or a configuration) of the game $s \in S$ is the vector of all the strategies of the players, $s = (s^1, \dots, s^n)$. Notice that for a given strategy profile s, several players can use paths that go through the same edge. For each edge $e \in E$ and a positive integer h, we have a cost $c_e(h) \in \mathbb{R}$ of the edge e for each player who uses a path containing e, provided that exactly h players share e. With each player i, we associate the cost function c^i mapping a strategy profile $s \in S$ to real numbers, i.e., $c^i : S \to \mathbb{R}$. For a strategy profile $s \in S$, let $n_e(s)$ be the number of players using the edge e in s. Then the cost the i-th player has to pay is

$$c^i(s) = \sum_{e \in E(P_i)} c_e(n_e(s)),$$

and the total cost of s is

$$c(s) = \sum_{i=1}^{n} c^i(s).$$

Rosenthal [21] proposed the study of the following potential function as a way of showing that a wide class of noncooperative games posess pure Nash equilibria. A *potential* of a strategy profile $s \in S$, or equivalently, the set of paths $(P_1, \dots, P_n)$, is defined as:

$$\Phi(s) = \sum_{e \in \cup_{i=1}^{n} E(P_i)} \sum_{h=1}^{n_e(s)} c_e(h). \tag{1}$$

In this paper, we are especially interested in the case where the cost of every edge is split evenly between the players sharing it, i.e, the payment of player i for edge e is $c_e(h) = \frac{c_e}{h}$ for $c_e \in \mathbb{R}$. Respectively, *Rosenthal potential* of a strategy profile $s \in S$ is

$$\Phi(s) = \sum_{e \in \cup_{i=1}^{n} E(P_i)} c_e \cdot \mathcal{H}(n_e(s)),$$

where $\mathcal{H}(h) = 1 + 1/2 + 1/3 + \cdots + 1/h$ is the h-th Harmonic number.

For a strategy profile $s \in S$ and $i \in \{1, 2, \dots, n\}$, we denote by s^{-i} the strategy profile of the players $j \neq i$, i.e. $s^{-i} = (s^1, \dots, s^{i-1}, s^{i+1}, \dots s^n)$. We use $(s^{-i}, \bar{s}^i)$ to denote the strategy profile identical to s, except that the ith player uses strategy $\bar{s}^i$ instead of s^i. Similarly, for a subset of players Π_0, we define $s^{-\Pi_0}$, the profile of players $j \notin \Pi_0$. For a strategy profile σ of players in Π_0, i.e., for an element of

Cartesian product of S^i for $i \in \Pi_0$, we denote by $(s^{-\Pi_0}, \sigma)$ the strategy profile of n players obtained from s by changing the strategies of players in Π_0 to σ.

A strategy profile $s \in S$ is a *Nash equilibrium* if no player $i \in \Pi$ can benefit from unilaterally deviating from his action to another action, i.e.,

$$\forall i \in \Pi \text{ and } \forall \bar{s}^i \in S^i,\ c^i(s^{-i}, \bar{s}^i) \geq c^i(s).$$

The crucial property of Rosenthal potential Φ is that each step performed by a player improving his payoff also decreases Φ (see [16, 21]). Consequently, if Φ admits a minimum value in a strategy profile, this strategy profile is a Nash equilibrium.

We say that a strategy profile s^* is *optimal* if it gives the minimum value of the potential, i.e., for any other strategy profile s, $\Phi(s) \geq \Phi(s^*)$.

Parameterized complexity. We briefly review the relevant concepts of parameterized complexity theory that we employ. For deeper background on the subject see the books by Downey and Fellows [7], Flum and Grohe [12], and Niedermeier [20].

In the classical framework of P *vs* NP, there is only one measurement (the overall input size) that frames the distinction between efficient and inefficient algorithms, and between tractable and intractable problems. Parameterized complexity is essentially a two-dimensional sequel, where in addition to the overall input size n, a secondary measurement k (the *parameter*) is introduced, with the aim of capturing the contributions to problem complexity due to such things as typical input structure, sizes of solutions, goodness of approximation, etc. Here, the parameter is deployed as a measurement of the amount of current solution modification allowed in a local search step. The parameter can also represent an aggregate of such bounds.

The central concept in parameterized complexity theory is the concept of *fixed-parameter tractability* (FPT), that is solvability of the parameterized problem in time $f(k) \cdot n^{O(1)}$. The importance is that such a running time isolates all the exponential costs to a function of the parameter only.

The main hierarchy of parameterized complexity classes is

$$FPT \subseteq W[1] \subseteq W[2] \subseteq \cdots \subseteq W[P] \subseteq XP.$$

The formal definition of classes $W[t]$ is technical, and, in fact, irrelevant to the scope of this paper. For our purposes it suffices to say that a problem is in a class if it is FPT-reducible to a complete problem in this class. Given two parameterized problems Π and Π', an *FPT reduction* from Π to Π' maps an instance (I, k) of Π to an instance (I', k') of Π' such that

(1) $k' = h(k)$ for some computable function h,
(2) (I, k) is a YES-instance of Π if and only if (I', k') is a YES-instance of Π', and
(3) the mapping can be computed in FPT time.

Hundreds of natural problems are known to be complete for the aforementioned classes, and $W[1]$ is considered the parameterized analog of NP, because the k-Step Halting Problem for nondeterministic Turing machines of unlimited nondeterminism (trivially solvable by brute force in time $O(n^k)$) is complete for $W[1]$. Thus, the statement $FPT \neq W[1]$ serves as a plausible complexity assumption for proving intractability results in parameterized complexity. INDEPENDENT SET, parameterized by solution size, is a more combinatorial example of a problem complete for $W[1]$. We refer the interested reader to the books by Downey and Fellows [7] or Flum and Grohe [12] for a more detailed introduction to the hierarchy of parameterized problems.

Local Search. Local search algorithms are among the most common heuristics used to solve computationally hard optimization problems. The common method of local search algorithms is to move from solution to solution by applying local changes. Books [1, 19] provide a nice introduction to the wide area of local search. Recall that the k-exchange neighborhood of a strategy profile s is the set of all profiles that can be obtained from s by changing strategies of at most k players, and the best response is the strategy (or strategies) which produces the most favorable outcome for a player, taking other players' strategies as given. Respectively, the best response dynamic is the following process. We start from an arbitrary strategy profile s. Then each player i in turn is given a possibility to modify his strategy to decrease the cost that he has to pay. We repeat this, giving all players a chance to change. We stop once we go through an entire round of players and nobody wants to change. Because each step performed by a player improving his payoff also decreases the Rosenthal potential Φ, the best-response dynamics in congestion games corresponds to local search in 1-exchange neighborhood minimizing Rosenthal potential Φ.

For two strategy profiles $s_1, s_2 \in S$, we define the Hamming distance $D(s_1, s_2) = |s_1 \triangle s_2|$ between s_1 and s_2, that is the number of players implementing different strategies in s_1 and s_2. We define *arena* as a directed graph G with root vertex r, a multiset of target vertices $t_1, \ldots, t_\ell$ and for every edge e of the graph a cost function $c_e : \mathbb{Z}^+ \to \mathbb{R}^+ \cup \{0\}$ such that $c_e(h) \geq c_e(h+1)$ for $h \geq 1$. We study the following parameterized version of the local search problem for multicast.

> p-LOCAL SEARCH ON POTENTIAL Φ **Parameter:** k
> **Input:** An arena consisting of graph G, vertices $r, (t_1, \ldots, t_\ell)$ and cost functions c_e, a set of agents Π with a strategy profile s, and an integer $k \geq 0$
> **Problem:** Decide whether there is a strategy profile s' such that $\Phi(s') < \Phi(s)$ and $D(s, s') \leq k$, where Φ is as defined in (1).

3 Minimizing Rosenthal Potential

The aim of this section is to prove the following theorem.

Theorem 3.1 *The p-LOCAL SEARCH ON POTENTIAL Φ problem is solvable in time*

$$\binom{|\Pi|}{k} \cdot 3^k \cdot |G|^{O(1)}.$$

Let us remark that in particular, if Φ is Rosenthal's potential, and hence the cost functions are of the special type $c_e(h) = \frac{c_e}{h}$, the p-LOCAL SEARCH ON POTENTIAL Φ problem can be solved within the running time of Theorem 3.1.

Recall that a strategy profile s^* is *optimal* if it gives the minimum value of the potential, i.e., for any other strategy profile s, $\Phi(s) \geq \Phi(s^*)$. If edge-sharing is profitable, then we can make the following observation about the structure of optimal strategies. Let G be a directed graph. Let also $s = (P_1, \dots, P_{|\Pi|})$ be a strategy profile. We say that s uses the arcs $\cup_{i=1}^{|\Pi|} E(P_i)$, and for a positive integer C, s uses C arcs if the union T of the paths P_i contains exactly C arcs.

Lemma 3.2 *Let C be an integer such that there is a strategy profile using at most C arcs. Let $s = (P_1, \dots, P_{|\Pi|})$ be a strategy profile using at most C arcs such that*

(i) Among all profiles using at most C arcs, s is optimal. In other words, for any profile s' using at most C arcs, we have $\Phi(s') \geq \Phi(s)$.
(ii) Subject to (i), s uses the minimum number of arcs.

Then the union T of the paths P_i, $i \in \{1, \dots, |\Pi|\}$, is an out-tree rooted in r.

Proof Targeting towards a contradiction, let us assume that $T = \cup_{i=1}^{|\Pi|} P_i$ is not an out-tree. Then there are paths $P_i, P_j, i, j \in \{1, \dots, |\Pi|\}$, that have a common vertex $v \neq r$ such that the (r, v)-subpaths P_i^v and P_j^v of P_i and P_j respectively enter v by different arcs.

We show first that

$$\sum_{e \in E(P_i^v)} c_e(n_e(s)) > \sum_{e \in E(P_j^v)} c_e(n_e(s)). \tag{2}$$

cannot occur. Assume that (2) holds. We claim that then the i-th player can improve his strategy and, consequently, Φ can be decreased, which will contradict the optimality of s. Denote by P the (r, t_i)-walk obtained from P_i by replacing path P_i^v by P_j^v. Notice that P is not necessarily a path as P_j^v can contain vertices and arcs of the (v, t_i)-subpath of P_i. Let w be the first vertex of P_j^v that is a vertex of the (v, t_i)-subpath of P_i. We denote by P' the (r, t_i)-path obtained by the concatenation of the (r, w)-subpath of P_j^v and the (w, t_i)-subpath of P_i. Notice that the set of arcs used by P' is a subset of arcs used by P, i.e., P' is obtained from P by possibly removing some loops. Let the strategy profile $s' = (s^{-i}, P')$. This profile uses ares that were used by s. Hence, it uses at most C arcs. By non-negativity of $c_e(h)$, the new cost for the i-th player is equal to

$$\begin{aligned}\sum_{e \in E(P')} c_e(n_e(s')) &= \sum_{e \in E(P') \cap E(P_i)} c_e(n_e(s)) + \sum_{e \in E(P') \setminus E(P_i)} c_e(n_e(s) + 1) \\ &\leq \sum_{e \in E(P) \cap E(P_i)} c_e(n_e(s)) + \sum_{e \in E(P) \setminus E(P_i)} c_e(n_e(s) + 1).\end{aligned}$$

Since for each $e \in E$ and $h \geq 1$, we have $c_e(h) \geq c_e(h+1)$,

$$\sum_{e \in E(P) \setminus E(P_i)} c_e(n_e(s)+1) \leq \sum_{e \in E(P) \setminus E(P_i)} c_e(n_e(s)).$$

Therefore,

$$\sum_{e \in E(P')} c_e(n_e(s')) \leq \sum_{e \in E(P)} c_e(n_e(s)).$$

By (2), we have

$$\sum_{e \in E(P)} c_e(n_e(s)) < \sum_{e \in E(P_i)} c_e(n_e(s)),$$

and the claim that player i can improve follows.

Hence,

$$\sum_{e \in E(P_i^v)} c_e(n_e(s)) \leq \sum_{e \in E(P_j^v)} c_e(n_e(s)).$$

By the same arguments as above, we can replace P_j by a (r, t_j)-path P in the walk obtained from P_j by the replacement of P_j^v by P_i^v without increasing Φ. Notice that s can have many paths that enter v by arcs that are different from the arc in P_i and, in particular, many paths can enter v via the same arc as P_j, But then we repeat the described operation for each path P_h in s with with this property. The modified strategy uses only arcs that were used in s. Therefore, this is a strategy profile that uses at most C arcs, but at least one arc that enters v is not used. It contradicts the choice of s. Hence, T is an out-tree rooted in r. □

We use Lemma 3.2 to find an optimal strategy profile using the approach proposed by Dreyfus and Wagner [8] for the STEINER TREE problem.

Theorem 3.3 *Given an arena as input, the minimum value of a potential Φ can be found in time $3^{|\Pi|} \cdot |G|^{O(1)}$. The algorithm can also construct the corresponding optimal strategy profile s^* within the same time complexity.*

Proof We give a dynamic programming algorithm. For simplicity, we only describe how to find the minimum of Φ, but it is straightforward to modify the algorithm to obtain the corresponding strategy profile.

Let $T = \{t_1, \ldots, t_{|\Pi|}\}$ be the multiset of terminals. We construct partial solutions for subsets $X \subseteq T$. Also, while at the end we are interested in the answer for the source r, our partial solutions are constructed for all vertices of G. For a vertex $u \in V(G)$ and a multiset $X \subseteq T$, let Γ_u^X denote the version of the game, in which only players associated with X build paths from u to their respective terminals. Therefore, we are interested in the game Γ_r^T. For a non-negative integer m, we define $\Psi(u, X, m)$ as the minimum value of the potential $\Phi(s)$ in the game Γ_u^X, taken over all strategy profiles s such that the union of paths in s contains at most m arcs (we say that s *uses* arc e if it is contained in some path from s). We assume that $\Psi(u, X, m) = +\infty$ if there are no feasible strategy profiles. Notice that by Lemma 3.2, the number of arcs used in an optimal strategy in the original problem is at most $|V(G)| - 1$. Hence, our aim is to compute $\Psi(r, T, |V(G)| - 1)$.

Clearly, $\Psi(u, \emptyset, m) = 0$ for all $u \in V$ and $m \geq 0$. For non-empty X and $m = 0$, $\Psi(u, X, 0) = 0$ if all terminals in X are equal to u, and $\Psi(u, X, 0) = +\infty$ otherwise. We need the following claim.

Claim 1 *For $X \neq \emptyset$ and $m \geq 1$, $\Psi(u, X, m)$ satisfies the following equation*:

$$\Psi(u, X, m) = \min\{\ \Psi(u, X, m-1),\\ \Psi(u, X \setminus Y, m_1) + \Psi(v, Y, m_2) + \sum_{h=1}^{|Y|} c_{(u,v)}(h)\}, \tag{3}$$

where the minimum is taken over all arcs $(u, v) \in E(G)$, $\emptyset \neq Y \subseteq X$, and $m_1, m_2 \geq 0$ such that $m_1 + m_2 = m - 1$; it is assumed that $\Psi(u, X, m) = \Psi(u, X, m-1)$ if the out-degree of u is zero.

Proof Let

$$\psi = \min\{\Psi(u, X, m-1), \Psi(u, X \setminus Y, m_1) + \Psi(v, Y, m_2) + \sum_{h=1}^{|Y|} c_{(u,v)}(h)\}.$$

We prove that $\Psi(u, X, m) = \psi$ by first showing that $\Psi(u, X, m) \geq \psi$, and then that $\Psi(u, X, m) \leq \psi$. Without loss of generality assume that $X = \{t_1, \ldots, t_\ell\} \subseteq T$, where $\ell = |X|$.

If $\Psi(u, X, m) = +\infty$, then $\Psi(u, X, m) \geq \psi$. Suppose that $\Psi(u, X, m) \neq +\infty$ and consider a strategy $s^* = (P_1, \ldots, P_\ell)$ in the game Γ_u^X which is optimal among those using at most m arcs and, subject to this condition, the number of used arcs is minimum; in particular, s^* has potential $\Psi(u, X, m)$. By Lemma 3.2, $H = \cup_{i=1}^{\ell} P_i$ is an out-tree rooted in u. If $|E(H)| < m$, then $\Psi(u, X, m) = \Psi(u, X, m-1) \geq \psi$. Assume that $|E(H)| = m$. As $m \geq 1$, vertex u has an out-neighbor v in H. Denote by H_1 and H_2 the components of $H - (u, v)$, where H_1 is an out-tree rooted in u and H_2 is an out-tree rooted in v. Let $Y \subseteq X$ be the multiset of terminals in H_2 and let $m_1 = |E(H_1)|$, $m_2 = |E(H_2)|$. Notice that exactly $|Y|$ players are using the arc (u, v) in s^* and Y is nonempty. Then $\Psi(u, X, m) \geq \Psi(u, X \setminus Y, m_1) + \Psi(v, Y, m_2) + \sum_{h=1}^{|Y|} c_{(u,v)}(h) \geq \psi$.

Now we prove that $\Psi(u, X, m) \leq \psi$. If $\psi = \Psi(u, X, m-1)$ then the claim is trivial, so let v, Y, m_1 and m_2 be such that $\psi = \Psi(u, X \setminus Y, m_1) + \Psi(v, Y, m_2) + \sum_{h=1}^{|Y|} c_{(u,v)}(h)$. Assume without loss of generality that $Y = \{t_1, \ldots, t_{\ell'}\}$ for some $\ell' \leq \ell$. If $\Psi(u, X \setminus Y, m_1) = +\infty$ or $\Psi(v, Y, m_2) = +\infty$, then the inequality is trivial. Suppose that $\Psi(u, X \setminus Y, m_1) \neq +\infty$ and $\Psi(v, Y, m_2) \neq +\infty$. Consider a strategy s_1^* in the game $\Gamma_u^{X \setminus Y}$ that is optimal among those using at most m_1 arcs, and a strategy s_2^* in the game Γ_v^Y that is optimal among those using at most m_2 arcs. Of course, the potential of s_1^* is equal to $\Psi(u, X \setminus Y, m_1)$, while the potential of s_2^* is equal to $\Psi(u, Y, m_2)$. We construct the strategy profile s in the game Γ_u^X as follows. For each terminal $t_j \in X \setminus Y$, the players use the (u, t_j)-path from s_1^*. For any $t_j \in Y$, the players use the (v, t_j)-path from s_2^* after accessing v from u via the arc (u, v), unless u already lies on this (v, t_j)-path, in which case they simply use the corresponding subpath of the (v, t_j)-path. Note that s uses at most $m_1 + m_2 + 1 = m$ arcs. Because for every $e \in E(G)$ and every $h \geq 1$, we have that $c_e(h) \geq 0$, and $c_e(h) \geq c_e(h+1)$, we infer that $\Phi(s) \leq \psi$, as possible overlapping of arcs used in s_1^*, s_2^* and the arc (u, v) can only decrease the potential of s. Since $\Psi(u, X, m) \leq \Phi(s)$, this implies that $\Psi(u, X, m) \leq \psi$. □

In order to finish the proof of Theorem 3.3, we need to show that using the recurrence (3) one can compute the value $\Psi(r, T, |V(G)| - 1)$ in time $3^{|\Pi|} \cdot |G|^{O(1)}$. The initial assignment for $\Psi(u, X, m)$ for the cases $m = 0$ or $X = \emptyset$ can be done in time $2^{|\Pi|} \cdot |G|^{O(1)}$ because we have $2^{|\Pi|}$ subsets X of T. Given the table of values of $\Psi(u, X, m - 1)$ for all $X \subseteq T$, we can compute the next table using (3) in time $3^{|\Pi|} \cdot |G|^{O(1)}$ because the number of pairs of sets (X, Y) such that $Y \subseteq X$ is 3^{Π}. Since the number of iterations is at most $|V(G)| - 1$, the total running time is $3^{|\Pi|} \cdot |G|^{O(1)}$.

We use Theorem 3.3 to construct algorithm for p-LOCAL SEARCH ON POTENTIAL Φ and to conclude with the proof of Theorem 3.1.

Proof of Theorem 3.1 Consider an instance of p-LOCAL SEARCH ON POTENTIAL Φ. Let $T = \{t_1, \ldots, t_{|\Pi|}\}$ be the multiset of terminals and let s be a strategy profile. Recall that p-LOCAL SEARCH ON POTENTIAL Φ asks whether at most k players can change their strategies in such a way that the potential decreases. Observe that we can assume that *exactly* k players are going to change their strategies because some of these players can choose their old strategies. There are $\binom{|\Pi|}{k}$ possibilities to choose a set of k players $\Pi_0 \subseteq \Pi$. We consider all possible choices and for each set Π_0, we check whether the players from this set can apply some strategy to decrease Φ.

Denote by $X \subseteq T$ the multiset of terminals of the players from Π_0, and let $s' = s^{-\Pi_0}$. We compute the potential $\Phi(s')$ for this strategy profile. Now we redefine the cost of edges as follows: for each $e \in E(G)$ and $h \geq 1$, $c'_e(h) = c_e(n_e(s') + h)$. Clearly, $c'_e(h) \geq 0$ and $c'_e(h) \geq c'_e(h + 1)$. Let Φ' be the potential for these edge costs. We find the minimum value of $\Phi'(s^*)$ for the set of players Π_0 and the corresponding terminals X. It remains to observe that $\Phi(s') + \Phi'(s^*) = \min\{\Phi(s'') \mid s'' = (s^{-\Pi_0}, \sigma),\ \sigma \in \prod_{i \in \Pi_0} S^i\}$. By Theorem 3.3, we can find $\Phi'(s^*)$ in time $3^k \cdot |G|^{O(1)}$ and the claim follows. □

4 Intractability of Local Search for Rosenthal Potential

This section is devoted to the proof of the following theorem.

Theorem 4.1 p-LOCAL SEARCH ON POTENTIAL Φ*, where* Φ *is Rosenthal potential for multicasting game, is W[1]-hard.*

Before we give the proof, let us remind a classical inequality that will be useful.

Definition 4.2. Let $a_1 \geq a_2 \geq \ldots \geq a_n$ and $b_1 \geq b_2 \geq \ldots \geq b_n$ be sequences of real numbers. We say that sequence (a_i) *majorizes* sequence (b_i), denoted $(a_i) \succeq (b_i)$, if $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$ and $\sum_{i=1}^k a_i \geq \sum_{i=1}^k b_i$ for all $1 \leq k < n$.

Theorem 4.3 (Hardy-Littlewood-Polyá inequality, [15]) *Let f be a convex function on interval $[a, b]$ and $a_1 \geq a_2 \geq \ldots \geq a_n$ and $b_1 \geq b_2 \geq \ldots \geq b_n$ be sequences of real numbers from $[a, b]$. If $(a_i) \succeq (b_i)$, then*

$$\sum_{i=1}^{n} f(a_i) \geq \sum_{i=1}^{n} f(b_i).$$

Let us note that by changing the sign of f we obtain that for concave functions the same result holds, but with inequality reversed.

We are now ready to prove Theorem 4.1.

Proof We provide an FPT reduction from the MULTICOLOURED CLIQUE problem, which is known to be W[1]-hard [11].

MULTICOLOURED CLIQUE	**Parameter:** k
Input: An undirected graph H with vertices partitioned into k sets $V_1, V_2, \ldots, V_k$, such that each set V_i is an independent set in H.	
Problem: Is there a clique C in H of size k?	

Observe that by the assumption that each V_i is an independent set, the clique C has to contain exactly one vertex from each part V_i.

We take an instance (H, k) of MULTICOLOURED CLIQUE and construct an instance $(G, s, k(k-1))$ of p-LOCAL SEARCH ON POTENTIAL Φ. First, we provide the construction of the new instance; then, we prove that the constructed instance is equivalent to the input instance of MULTICOLOURED CLIQUE. During the reduction we assume k to be large enough; for constant k we solve the instance (H, k) in polynomial time by a brute-force search and output a trivial YES or NO instance of p-LOCAL SEARCH ON POTENTIAL Φ.

Construction. First create the root vertex r. For every $u \in V_i$, we create k vertices: $\overline{u}$ and $u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_k$. Denote by F_u the set $\{u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_k\}$. We connect the created vertices in the following manner: we construct one arc $(r, \overline{u})$ with cost $R = k^2$, and for all $j \in \{1, 2, \ldots, i-1, i+1, \ldots, k\}$ we construct arc $(\overline{u}, u_j)$ with cost 0. With every vertex u_j for all $u \in V(H)$ we associate a player that builds a path from r to u_j. In the initial strategy profile s, each of $(k-1)|V(H)|$ players builds a path that leads to his vertex via the corresponding vertex $\overline{u}$. Observe that the potential of this strategy profile is equal to $|V(H)| \cdot R \cdot \mathcal{H}(k-1)$.

We now construct the part of the graph that is responsible for the choice of the clique. We create a *pseudo-root* r' and an arc (r, r') with cost

$$W = \frac{1}{\mathcal{H}(k(k-1))} \left(k \cdot R \cdot \mathcal{H}(k-1) - \frac{3}{2}\binom{k}{2} - \varepsilon \right),$$

where $\varepsilon = \frac{k-1}{k^5}$. The value of W is tailored to separate cliques in H from subgraphs that lack at least one edge using budget constraints; the meaning of every summands

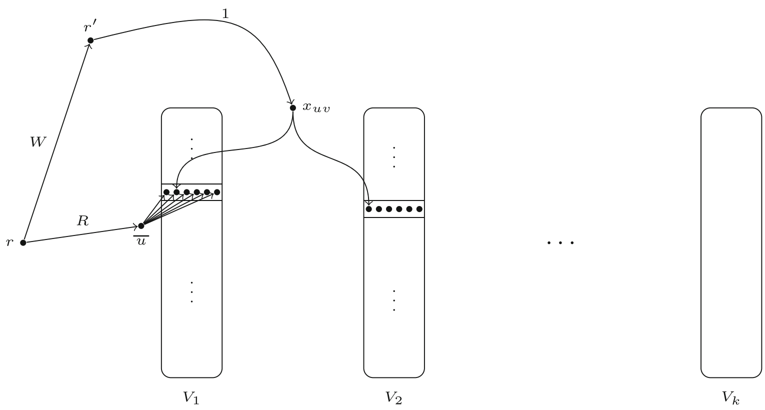

Fig. 1 Graph G

will become clear in the proof of correctness of the reduction. Note that $W \geq 1$ for sufficiently large k.

For every edge $uv \in E(H)$, where $u \in V_i$ and $v \in V_j$, $i \neq j$, we create a vertex x_{uv}, arc (r', x_{uv}) of cost 1, and arcs (x_{uv}, u_j), (x_{uv}, v_i) of cost 0. This concludes the construction shown in Fig. 1.

Before we proceed with the formal proof of the theorem, let us give some intuition behind the construction. Given a clique C in H, we can construct a common strategy of $k(k-1)$ players assigned to vertices from $\bigcup_{u \in V(C)} F_u$, who can agree to jointly rebuild their paths via the pseudo-root r'. The "cost of entrance" for remodelling the strategy in this manner is paying for the expensive arc (r, r'); however, this can amortised by sharing cheap arcs (r', x_{uv}) for $uv \in E(C)$. The costs have been chosen so that only the maximum possibility of sharing, which corresponds to a clique in H, can yield a decrease of the potential.

From a clique to a remodelled strategy profile. Assume that C is a clique in H with k vertices. Let us remind, that in the initial strategy profile s each player is using the corresponding arc $(r, \overline{u})$ for his path. We construct the new strategy profile s' by changing strategies of $k(k-1)$ players as follows. For every $uv \in E(C)$, where $u \in V_i$ and $v \in V_j$, $i \neq j$, the players associated with vertices u_j and v_i reroute their paths so that in s' they lead via r' and x_{uv} to respective targets. In comparison to the profile s, the new profile s':

- has congestion withdrawn from arcs $(r, \overline{u})$ for $u \in V(C)$—this decreases the potential by $k \cdot R \cdot \mathcal{H}(k-1)$;
- has congestion introduced to arcs (r, r') and (r', x_{uv}) for $uv \in E(C)$—this increases the potential by $W \cdot \mathcal{H}(k(k-1)) + \frac{3}{2}\binom{k}{2}$.

Therefore, $\Phi(s') = \Phi(s) - k \cdot R \cdot \mathcal{H}(k-1) + W \cdot \mathcal{H}(k(k-1)) + \frac{3}{2}\binom{k}{2} = \Phi(s) - \varepsilon < \Phi(s)$.

From a remodelled strategy profile to a clique. Recall that s is an initial strategy profile each player uses the corresponding arc $(r, \overline{u})$ for his path. Let s' be a strategy profile such that $\Phi(s') < \Phi(s)$ and $D(s, s') = \ell \leq k(k-1)$. Let L be the set of players who have rebuilt their strategies in s'; then $|L| = \ell$. Let p be a player in L, who is assigned to a vertex $u_j \in F_u$ for some $u \in V(H)$. Observe, that the only possibility of rebuilding the strategy for p is to choose a path leading through r' and a vertex x_{uv} for some $uv \in E(H)$, $v \in V_j$. We now examine all the arcs of the graph G with nonzero costs in order to provide a lower bound on $\Delta\Phi = \Phi(s') - \Phi(s)$. We partition the arcs into three classes: (i) arcs $(r, \overline{u})$ for $u \in V(H)$, (ii) arc (r, r'), and (iii) arcs (r', x_{uv}) for $uv \in E(H)$. For each of these classes we analyze the *contribution* to the difference $\Delta\Phi$; by this we mean the difference of contributions to potentials $\Phi(s')$ and $\Phi(s)$ from the corresponding arcs.

Firstly, consider arcs $(r, \overline{u})$ for $u \in V(H)$. In total, ℓ players withdraw their paths from these arcs. The contribution to s' of these arcs is equal to $\sum_{u \in V(H)} R \cdot \mathcal{H}(a_{\overline{u}})$, where $a_{\overline{u}}$ is the number of players using the arc $(r, \overline{u})$ in strategy profile s'. We know that $\sum_{u \in V(H)} a_{\overline{u}} = (k-1)|V(H)| - \ell$, while $0 \leq a_{\overline{u}} \leq k-1$ for all $u \in V(H)$. Observe that then the sequence $(a_{\overline{u}})$ is majorized by a sequence consisting of $|V(H)| - \left\lfloor \frac{\ell}{k-1} \right\rfloor - 1$ terms $(k-1)$, one term $(k-1) - (\ell \bmod (k-1))$, and $\left\lfloor \frac{\ell}{k-1} \right\rfloor$ zeroes. Therefore, since $\mathcal{H}$ can be extended to a concave function, by Theorem 4.3 we infer that:

$$\sum_{u \in V(H)} \mathcal{H}(a_{\overline{u}}) \geq \left(|V(H)| - \left\lceil \frac{\ell}{k-1} \right\rceil \right) \mathcal{H}(k-1) + \mathcal{H}((k-1) - (\ell \bmod (k-1))). \tag{4}$$

Moreover, from concavity of function $\mathcal{H}$ we infer that

$$\mathcal{H}((k-1) - (\ell \bmod (k-1))) \geq \frac{(k-1) - (\ell \bmod (k-1))}{k-1} \mathcal{H}(k-1). \tag{5}$$

Using (4) and (5) we infer that

$$\sum_{u \in V(H)} \mathcal{H}(a_{\overline{u}}) \geq \left(|V(H)| - \frac{\ell}{k-1} \right) \mathcal{H}(k-1). \tag{6}$$

This implies that the contribution of these arcs to $\Delta\Phi$ is at least $K_1 = -R \cdot \frac{\ell}{k-1} \cdot \mathcal{H}(k-1)$.

Now, consider the arc (r, r'). There are exactly ℓ players using this arc in s', while in s nobody was using it. Therefore, the contribution from this arc to $\Delta\Phi$ is equal to $K_2 = W \cdot \mathcal{H}(\ell)$.

Finally, consider arcs (r', x_{uv}) for $uv \in E(H)$. All the ℓ players which rebuild their strategies in s' use exactly one such arc. Moreover, each of these edges can be shared by at most two players. Therefore, the contribution to $\Delta\Phi$ from these edges is at least $K_3 = \frac{\ell}{2} \cdot \mathcal{H}(2) = \frac{3}{4}\ell$, and the contribution is larger by at least $\frac{1}{2}$ if any player does not share the arc with some other player.

Concluding, since $\Phi(s') < \Phi(s)$ we have that

$$0 > \Delta\Phi \geq K_1 + K_2 + K_3 = \frac{3}{4}\ell + W \cdot \mathcal{H}(\ell) - R \cdot \frac{\ell}{k-1} \cdot \mathcal{H}(k-1).$$

Therefore,

$$R \cdot \frac{\mathcal{H}(k-1)}{k-1} > \frac{3}{4} + W \cdot \frac{\mathcal{H}(\ell)}{\ell}. \quad (7)$$

Here we used values K_1, K_2 and K_3 as lower bounds on the total contribution from respective classes of arcs. Note that if at some point we infer that the contribution from any of these classes is actually larger than the corresponding lower bound K_q, for $q \in \{1, 2, 3\}$, for instance because one of the arcs is contributing more than assumed in the presented estimations, then we can add a corresponding term to the right-hand side of equation (7).

We now prove three structural claims about the remodelled strategy s', which lead us to a conclusion that s' have to originate in a clique in H.

Claim 1 *It holds that* $\ell = k(k-1)$.

Proof Let us define $g(t) = \frac{\mathcal{H}(t)}{t}$. Observe that for $t > 1$ we have that

$$\begin{aligned} g(t) - g(t-1) &= \frac{\mathcal{H}(t)}{t} - \frac{\mathcal{H}(t-1)}{t-1} = \mathcal{H}(t-1)\left(\frac{1}{t} - \frac{1}{t-1}\right) + \frac{1}{t^2} \\ &= \frac{1}{t^2} - \frac{\mathcal{H}(t-1)}{t(t-1)} \le \frac{1}{t^2} - \frac{1}{t(t-1)} \le -\frac{1}{t^3}. \end{aligned}$$

Hence, function g is decreasing and $\frac{\mathcal{H}(\ell)}{\ell} \ge \frac{\mathcal{H}(k(k-1))}{k(k-1)}$.

Assume that $\ell < k(k-1)$; then it follows that $\frac{\mathcal{H}(\ell)}{\ell} \ge \frac{\mathcal{H}(k(k-1))}{k(k-1)} + \frac{1}{k^6}$. We obtain that

$$\begin{aligned} R \cdot \frac{\mathcal{H}(k-1)}{k-1} &> \frac{3}{4} + W \cdot \frac{\mathcal{H}(k(k-1))}{k(k-1)} + W \cdot \frac{1}{k^6} \\ &= \frac{3}{4} + \frac{1}{k(k-1)}\left(k \cdot R \cdot \mathcal{H}(k-1) - \frac{3}{2}\binom{k}{2} - \varepsilon\right) + W \cdot \frac{1}{k^6} \\ &= \frac{3}{4} + R \cdot \frac{\mathcal{H}(k-1)}{k-1} - \frac{3}{4} - \frac{\varepsilon}{k(k-1)} + W \cdot \frac{1}{k^6} \\ &= R \cdot \frac{\mathcal{H}(k-1)}{k-1} + \frac{W-1}{k^6} \ge R \cdot \frac{\mathcal{H}(k-1)}{k-1}. \end{aligned}$$

The last inequality follows from $W \ge 1$, which is true for k large enough. This contradiction shows that $\ell = k(k-1)$. □

Claim 2 *In strategy profile* s'*, every arc of the form* (r', x_{uv}) *is used by zero players or by exactly two players.*

Proof Obciously, every arc of the form (r', x_{uv}) can be used by at most two players. Now we want to prove that no arc (r', x_{uv}) can be used by exactly one player in the strategy profile s'. For the sake of contradiction, we assume that at least one of the arcs (r', x_{uv}) is used by exactly one player in s'. Then the total contribution to $\Delta\Phi$

of the arcs of the form (r', x_{uv}) is at least $\frac{3}{4}\ell + \frac{1}{2}$. Similarly as before, we obtain that

$$\begin{aligned} R \cdot \frac{\mathcal{H}(k-1)}{k-1} &> \frac{3}{4} + \frac{1}{2k(k-1)} + W \cdot \frac{\mathcal{H}(k(k-1))}{k(k-1)} \\ &= R \cdot \frac{\mathcal{H}(k-1)}{k-1} + \frac{1}{2k(k-1)} - \frac{\varepsilon}{k(k-1)} \\ &= R \cdot \frac{\mathcal{H}(k-1)}{k-1} + \left(\frac{1}{2} - \varepsilon\right) \cdot \frac{1}{k(k-1)} \geq R \cdot \frac{\mathcal{H}(k-1)}{k-1} \end{aligned}$$

This contradiction shows that in strategy profile s', all the arcs of the form (r', x_{uv}) are used by zero or by two players. □

Claim 3 *The sequence $(a_{\overline{u}})$ contains exactly $|V(H)| - k$ terms $k-1$ and k zeroes, i.e., the set of players that did rebuild their strategies is concentrated on k vertices of H.*

Proof Suppose that this is not the case. Then the sequence $(a_{\overline{u}})$ is majorized by a sequence containing $|V(H)|-k-1$ terms $k-1$, one term $k-2$, one term 1 and $k-1$ zeroes. By Theorem 4.3, the contribution of arcs of the form $(r, \overline{u})$ to $\Delta\Phi$ is at least

$$\begin{aligned} &R \cdot ((|V(H)| - k - 1) \cdot \mathcal{H}(k-1) + \mathcal{H}(k-2) + \mathcal{H}(1)) - R \cdot |V(H)| \cdot \mathcal{H}(k-1) \\ &\quad = R \cdot (|V(H)| - k) \cdot \mathcal{H}(k-1) + R \cdot \left(1 - \frac{1}{k-1}\right) - R \cdot |V(H)| \cdot \mathcal{H}(k-1) \\ &\quad = -R \cdot k \cdot \mathcal{H}(k-1) + R \cdot \left(1 - \frac{1}{k-1}\right). \end{aligned}$$

Similarly as before, we obtain that

$$\begin{aligned} R \cdot \frac{\mathcal{H}(k-1)}{k-1} &> \frac{3}{4} + W \cdot \frac{\mathcal{H}(k(k-1))}{k(k-1)} + \frac{1}{k(k-1)} \cdot R \cdot \left(1 - \frac{1}{k-1}\right) \\ &= R \cdot \frac{\mathcal{H}(k-1)}{k-1} + \frac{R}{k(k-1)} \cdot \left(1 - \frac{1}{k-1}\right) - \frac{\varepsilon}{k(k-1)} \\ &\geq R \cdot \frac{\mathcal{H}(k-1)}{k-1}. \end{aligned}$$

This contradiction proves the claim. □

Claim 3 shows that we can distinguish k vertices $u^1, u^2, \ldots, u^k$ such that L is exactly the set of players assigned to vertices $\bigcup_{i=1}^{k} F_{u^i}$. We claim that $H[\{u^1, u^2, \ldots, u^k\}]$ is a clique, which will conclude the proof.

Consider the vertex u^1. Without loss of generality we assume that $u^1 \in V_1$. By Claim 2, in strategy profile s', the player associated with vertex u^1_j has to share an arc $(r', x_{u^1 v})$ for some $v \in V_j$, for $j = 2, 3, \ldots, k$. Therefore, the set $\{u^2, \ldots, u^k\}$ has to contain a vertex from each of the sets $V_2, V_3, \ldots, V_k$. Assume then, without loss of generality, that $u^j \in V_j$ for all $j = 2, \ldots, k$.

Let us take u^i and u^j for $i \neq j$; we argue that $u^i u^j \in E(H)$, which will finish the proof. Consider players associated with vertices u^i_j and u^j_i. Again by Claim 2, they have to share an arc outgoing from r', so there has to exist a vertex $x_{u^i u^j}$ and the corresponding arc $(r', x_{u^i u^j})$. From the construction of G we infer that $u^i u^j \in E(H)$.

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