Kernels for (Connected) Dominating Set on Graphs with Excluded Topological Minors

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We give the first linear kernels for the Dominating Set and Connected Dominating Set problems on graphs excluding a fixed graph H as a topological minor. In other words, we prove the existence of polynomial time algorithms that, for a given H-topological-minor-free graph G and a positive integer k, output an H-topological-minor-free graph G' on $O(k)$ vertices such that G has a (connected) dominating set of size k if and only if G' has one.

Our results extend the known classes of graphs on which the Dominating Set and Connected Dominating Set problems admit linear kernels. Prior to our work, it was known that these problems admit linear kernels on graphs excluding a fixed apex graph H as a minor. Moreover, for Dominating Set, a kernel of size $k^{c(H)}$, where $c(H)$ is a constant depending on the size of H, follows from a more general result on the kernelization of Dominating Set on graphs of bounded degeneracy. Alon and Gutner explicitly asked whether one can obtain a linear kernel for Dominating Set on H-minor-free graphs. We answer this question in the affirmative and in fact prove a more general result. For Connected Dominating Set no polynomial kernel even on H-minor-free graphs was known prior to our work. On the negative side, it is known that Connected Dominating Set on 2-degenerated graphs does not admit a polynomial kernel unless coNP $\subseteq$ NP/poly.

Our kernelization algorithm is based on a non-trivial combination of the following ingredients

- The structural theorem of Grohe and Marx [STOC 2012] for graphs excluding a fixed graph H as a topological minor;
- A novel notion of protrusions, different than the one defined in [FOCS 2009];
- Our results are based on a generic reduction rule that produces an equivalent instance (in case the input graph is H-minor-free) of the problem, with treewidth $O(\sqrt{k})$. The application of this rule in a divide-and-conquer fashion, together with the new notion of protrusions, gives us the linear kernels.

A protrusion in a graph [FOCS 2009] is a subgraph of constant treewidth which is separated from the rest of the graph by at most a constant number of vertices. In our variant of protrusions, instead of stipulating that the subgraph be of constant *treewidth*, we ask that it contains a *constant number of vertices from a solution.*

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Fedor V. Fomin was supported by the European Research Council through ERC Grant Agreement n. 267959. Daniel Lokshtanov was supported by the Bergen Research Foundation and the University of Bergen through project "BeHard". Dimitrios M. Thilikos was co-financed by the E.U. (European Social Fund - ESF) and Greek national funds through the Operational Program "Education and Lifelong Learning" of the National Strategic Reference Framework (NSRF) - Research Funding Program: "Thales. Investing in knowledge society through the European Social Fund."
Preliminary versions of this paper appeared in SODA 2012 [42] and STACS 2013 [43].
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https://doi.org/10.1145/3155298

We believe that this new take on protrusions would be useful for other graph problems and in different algorithmic settings.

CCS Concepts: • **Theory of computation → Parameterized complexity and exact algorithms**;

Additional Key Words and Phrases: Kernelization, connected dominating set, topological minor free graphs

ACM Reference format:
Fedor V. Fomin, Daniel Lokshtanov, Saket Saurabh, and Dimitrios M. Thilikos. 2018. Kernels for (Connected) Dominating Set on Graphs with Excluded Topological Minors. *ACM Trans. Algorithms* 14, 1, Article 6 (January 2018), 31 pages.
https://doi.org/10.1145/3155298

1 INTRODUCTION

Kernelization is a well-established subarea of parameterized complexity. A parameterized problem is said to admit a *polynomial kernel* if there is a polynomial time algorithm (the degree of polynomial being independent of the parameter k), called a *kernelization* algorithm, that reduces the input instance down to an instance with size bounded by a polynomial $p(k)$ in k, while preserving the answer. This reduced instance is called a $p(k)$ *kernel* for the problem. If the size of the kernel is $O(k)$, then we call it a *linear kernel* (for a more formal definition, see Section 2). Kernelization has turned out to be an interesting computational approach both from practical and theoretical perspectives. There are many real-world applications where even very simple preprocessing can be surprisingly effective, leading to significant reductions in the size of the input. Kernelization is a natural tool not only for measuring the quality of preprocessing rules proposed for specific problems but also for designing new powerful preprocessing algorithms. From the theoretical perspective, kernelization provides a deep insight into the hierarchy of parameterized problems in FPT, the most interesting class of parameterized problems. There are also interesting links between lower bounds on the sizes of kernels and classical computational complexity [11, 19, 30].

The Dominating Set (DS) problem together with its numerous variants, is one of the most classical and well-studied problems in algorithms and combinatorics [49]. In the Dominating Set (DS) problem, we are given a graph G and a non-negative integer k, and the question is whether G contains a set of k vertices whose closed neighborhood contains all the vertices of G. The connected variant of the problem, Connected Dominating Set (CDS) asks, given a graph G and a non-negative integer k, whether G contains a dominating set D of at most k vertices such that for every connected component C of G, we have that $G[V(C) \cap D]$ is connected. This definition of CDS differs slightly from the established one where one just demands that the subgraph induced by the dominating set be connected. Our definition generalizes the established one to include disconnected graphs. A considerable part of the algorithmic study of these NP-complete problems has been focused on the design of parameterized and kernelization algorithms. In general, DS is W[2]-complete and therefore it cannot be solved by a parameterized algorithm, unless an unexpected collapse occurs in the Parameterized Complexity hierarchy (see References [27, 36, 55]) and thus also does not admit a kernel. However, there are interesting graph classes where *fixed-parameter tractable* (FPT) algorithms exist for the DS problem. The project of widening the families of graph classes, on which such algorithms exist, inspired a multitude of ideas that made DS the test bed for some of the most cutting-edge techniques of parameterized algorithm design. For example, the initial study of parameterized subexponential algorithms for DS on planar graphs [2, 20, 44] resulted in the creation of bidimensionality theory characterizing a broad range of graph problems that admit efficient approximation schemes, fixed-parameter algorithms, or kernels on a broad range of graphs [21, 23, 26, 39–41].

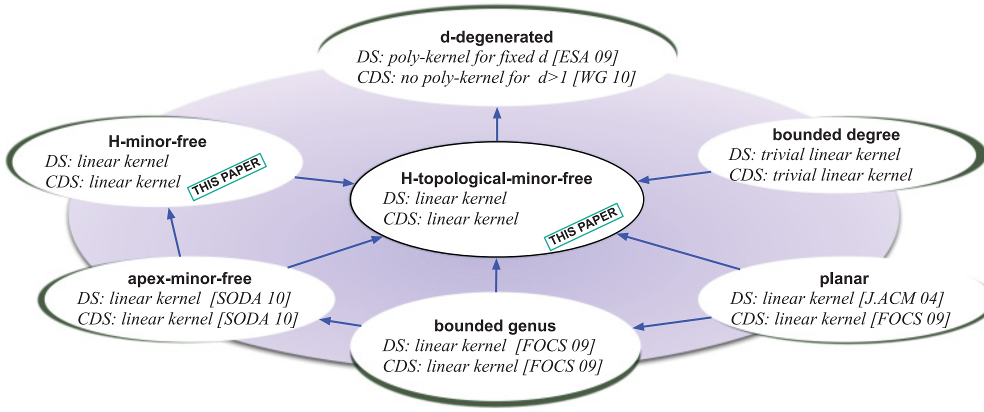

Fig. 1. Kernels for DS and CDS on classes of sparse graphs. Arrows represent inclusions of classes. In the diagram, [J.ACM 04] refers to Albers et al. [3], [FOCS 09] to the Bodlaender et al. [12], [SODA 10] and [SODA 12] to the Fomin et al. [41, 42], [ESA 09] to the Philip et al. [56], and [WG 10] to the Cygan et al. [17].

One of the first results on linear kernels is the celebrated work of Alber et al. on DS on planar graphs [3]. This work augmented significantly the interest in proving polynomial (or preferably linear) kernels for other parameterized problems. The result of Alber et al. [3], see also Reference [15], has been extended to much more general graph classes like graphs of bounded genus [12] and apex-minor-free graphs [41]. An important step in this direction was made by Alon and Gutner [4, 48], who obtained a kernel of size $O(k^h)$ for DS on H-minor-free and H-topological-minor-free graphs, where the constant h depends on the excluded graph H. Later, Philip et al. [56] obtained a kernel of size $O(k^h)$ on $K_{i,j}$-free and d-degenerate graphs, where h depends on i, j, and d respectively. In particular, for d-degenerate graphs, a subclass of $K_{i,j}$-free graphs, the algorithm of Philip et al. [56] produces a kernel of size $O(k^{d^2})$. Similarly, the sizes of the kernels in References [4, 48, 56] are bounded by polynomials in k with degrees depending on the size of the excluded minor H. Alon and Gutner [4] mentioned as a challenging question whether one can characterize the families of graphs for which the dominating set problem admits a linear kernel, that is, a kernel of size $f(h) \cdot k$, where the function f depends *exclusively* on the graph family. In this direction, there are already results for more restricted graph classes. According to the meta-algorithmic results on kernels introduced in Reference [12], DS has a kernel of size $f(g) \cdot k$ on graphs of genus g. An alternative meta-algorithmic framework, based on bidimensionality theory [21], was introduced in Reference [41], implying the existence of a kernel of size $f(H) \cdot k$ for DS on graphs excluding an apex[1] graph H as a minor. While apex-minor-free graphs form much more general class of graphs than graphs of bounded genus, H-minor-free graphs and H-topological-minor-free graphs form much larger classes than apex-minor-free graphs. For example, the class of graphs excluding $H = K_6$, the complete graph on six vertices, as a minor, contains all apex graphs. Alon and Gutner [4] and Gutner [48] posed as an open problem whether one can obtain a linear kernel for DS on H-minor-free graphs. Prior to our work, the only result on linear kernels for DS on graphs excluding a fixed graph H as a topological minor was the result of Alon and Gutner [4] for the special case where $H = K_{3,h}$. See Figure 1 for the relationship between these classes.

It is tempting to conjecture that similar improvements on kernel sizes are possible for more general graph classes like d-degenerate graphs. For example, for graphs of bounded vertex degree,

[1]An *apex* graph is a graph that can be made planar by the removal of a single vertex.

a subclass of d-degenerate graphs, DS has a trivial linear kernel. Unfortunately, for d-degenerate graphs the existence of a linear kernel, or even a polynomial kernel with the exponent of the polynomial being independent of d, is very unlikely. By the recent work of Cygan et al. [16], the kernelization algorithm of Philip et al. [56] is essentially tight—the existence of a kernel of size $O(k^{(d-3)(d-1)-\varepsilon})$ for DS on d-degenerate graphs would imply that coNP is contained in NP/poly.

In this work, we show how to generalize the linearity of kernelization for DS from bounded-degree graphs and apex minor free graphs to the class of graphs excluding a fixed graph H as a topological minor. Moreover, a modification of the ideas for DS kernelization can be used to obtain a linear kernel for CDS, which is usually a much more difficult problem to handle due to the connectivity constraint. For example, CDS does not have a polynomial kernel on 2-degenerate graphs unless coNP is in NP/poly [17]. We must *emphasize* that our linear kernels are existential. That is, we just show the mere existence of polynomial time algorithms computing linear kernels.

The class of graphs excluding a fixed graph H as a topological minor is a wide class of graphs containing H-minor-free graphs and graphs of constant vertex degrees. The existence of a linear kernel for DS on this class of graphs significantly extends and improves previous works [4, 42, 48]. The extension of the results for planar graphs from Reference [3] and apex-minor-free graphs from Reference [41] to the more general family of H-minor-free graphs requires several new ideas. Similar difficulties in generlizing algorithmic techniques from apex-minor free to H-minor-free graphs were observed in approximation [24] and parameterized algorithms [21, 28]. The basic idea behind kernelization algorithms on apex-minor-free graphs is the bidimensionality of DS. Roughly speaking, the treewidth of these graphs with dominating set of size k is $o(k)$. In other words, excluding an apex graph makes it possible to bound the tree-decomposability of the input graph by a *sublinear* function of the size of a dominating set, which is not the case for more general classes of H-minor-free graphs or a family of graphs excluding a fixed graph H as a topological minor.

A main ingredient of our kernelization algorithms are new reduction rules that allow us to obtain the desired kernels on H-minor-free graphs. This is an important step for our kernel on the class of graphs excluding a fixed graph H as a topological minor. The main idea behind our algorithm is to identify and remove "irrelevant" vertices without changing the solution such that in the reduced graph one can select $O(k)$ vertices whose removal leaves protrusions, that is, subgraphs of constant treewidth separated from the remaining vertices by a constant number of vertices. If we are able to obtain such a graph, then we can use the techniques from Reference [41] to construct the linear kernel. Roughly speaking, our rule to identify "irrelevant" vertices works as follows: We try specific vertex subsets of constant size, and, for each subset, we try all "feasible" scenarios for how dominating sets can interact with the subset and find neighbours of theses subsets whose removal does not change the outcome of any feasible scenario. The main difference of this new reduction rule in comparison to other rules for DS [3, 15] is that instead of reducing the size of the graph to $O(k)$, it reduces the treewidth of the graph to $O(\sqrt{k})$. Thus ideawise, it is closer to the "irrelevant vertex" approach developed by Robertson and Seymour for disjoint paths and minor checking problems [57]. However, the significant difference with this technique is that in all applications of "irrelevant vertex" the bounds on the treewidth are exponential or even worse [51, 52, 54]. Moreover, Adler et al. [1] provide instances of the disjoint paths problem on planar graphs, for which the irrelevant vertex approach of Robertson and Seymour produces graphs of treewidth $2^{\Omega(k)}$. Our rule provides a reduced graph with *sublinear* treewidth for DS.

The proof that after deletion of all irrelevant vertices the treewidth of the graph becomes sublinear is non-trivial. For this proof, we need the theorem of Robertson and Seymour [58] on decomposing a graph into a set of torsos connected via clique-sums. By making use of this theorem, we show that, by applying the rule for all subsets of apex vertices of each torso, it is possible to reduce the treewidth of each torso to $O(\sqrt{k})$. This implies that the treewidth of the reduced graph is also

$\mathcal{O}(\sqrt{k})$. However, the number of torsos can be $\Omega(n)$ and the sublinear treewidth of the reduced graph still does not bring us directly to the kernel. To overcome this obstacle, we have to implement the irrelevant vertex rule in a divide-and-conquer manner, and only after doing this can we guarantee that the reduced graph admits a linear kernel. The idea of using divide and conquer in kernelization is our first conceptual contribution.

The second main step of our kernelization algorithm for DS, on the class of graphs excluding a fixed graph H as a topological minor, is to design reduction rules for graphs of bounded degree. The ideas introduced for H-minor-free graphs can hardly work on graphs of bounded degree and hence on graphs excluding a fixed graph H as a topological minor. The reason is that the bound $o(k)$ on the treewidth of such graphs would imply that DS is solvable in subexponential time on graphs of bounded degree, which in turn can be shown to contradict the Exponential Time Hypothesis [50]. This is why the kernelization techniques developed for H-minor-free graphs do not seem to be applicable directly in our case.

High Level Overview of the Main Ideas. Our kernelization algorithm has two main phases. In the first phase, we partition the input graph G into subgraphs $C_0, C_1, \ldots, C_\ell$, such that $|C_0| = \mathcal{O}(k)$; for every $i \geq 1$, the neighbourhood $N(C_i) \subseteq C_0$, and $\sum_{1 \leq i \leq \ell} |N(C_i)| = \mathcal{O}(k)$. In the second phase, we replace these graphs by smaller equivalent graphs. Towards this, we treat graphs $N[C_i] = C_i \cup N(C_i)$, $i \geq 1$, as t-boundaried graphs with boundary $N(C_i)$. Our second conceptual contribution is a polynomial time algorithmic procedure for replacing a t-boundaried graph by an equivalent graph of size $\mathcal{O}(|N(C_i)|)$. Observe that as a result of such replacements, the size of the new graph is

$$\sum_{1 \leq i \leq \ell} |\mathcal{O}(N(C_i))| + |C_0| = \mathcal{O}(k),$$

and thus we obtain a linear kernel. Kernelization techniques based on replacing a t-boundaried graph by an equivalent instance or, more specifically, protrusion replacement were used before in References [12, 38, 41, 53]. At this point, it is also important to mention earlier works [7, 13, 14, 18, 35] on protrusion replacement in the algorithmic setting on graphs of bounded treewidth. The substantial differences with our replacement procedure and the ones used before in the kernelization setting are the following:

- In the protrusion replacement procedure, it is assumed that the size of the boundary t and the treewidth of the replaced graph are constants. In our case, neither the treewidth nor the boundary size are bounded. In particular, the boundary size could be a *linear* function of k.
- In earlier protrusion replacements, the size of the equivalent replacing graph is bounded by some (non-elementary) function of t. In our case, this is a *linear* function of t.

Our new replacement procedure strongly exploits the fact that graphs C_i possess a set of desired properties allowing us to apply the irrelevant vertex technique explained earlier. However, not every graph G excluding some fixed graph as a topological minor can be partitioned into graphs with the desired properties. We show that, in this case, there is another polynomial time procedure transforming G into an equivalent graph, which in turn can be partitioned. The procedure is based on a generalized notion of protrusion, which is the third conceptual contribution of this article. In the new notion of protrusion, we relax the requirement that protrusions are of bounded treewidth by the condition that they have a bounded size dominating set. Let us note that a similar notion of a generalized protrusion, bounded by the size of a certificate, can be used for a variety of graph problems. We show that either a graph does not have the desired partition or it contains a sufficiently large generalized protrusion, which can be replaced by a smaller equivalent subgraph.

The construction of the partitioning is heavily based on the recent work of Grohe and Marx on the structure of such graphs [47].

As a by-product of our results, we obtain the first subexponential time algorithms for Connected Dominating Set, a deterministic algorithm solving the problem on an n-vertex H-minor-free graph in time $2^{O(\sqrt{k})} + n^{O(1)}$. For Dominating Set, our results imply a significant simplification and refinement of a $2^{O(\sqrt{k})} n^{O(1)}$ algorithm on H-minor-free graphs due to Demaine et al. [21]. Also our kernels can be used to obtain, subexponential, polynomial-space parameterized algorithms for these problems.

Organization of the Article. The remaining part of this article is organized as follows. In Section 2, we provide definitions and state known results used in the article. In Section 3, we introduce the new notion of "generalized protrusions" and build a theory of replacements for such protrusions. We provide a decomposition lemma in Section 4, which will be used for kernelization algorithms. In Sections 5 and 6, we give the two main results of the article, linear kernels for DS and CDS on the class of graphs excluding a fixed graph H as a topological minor. In Section 7, we conclude with questions for further research and give a short overview of some of the developments that have happened since the conference versions of this article were published, including work on kernelization of DS and CDS on graphs of bounded expansion and on nowhere-dense graphs.

2 PRELIMINARIES

In this section, we give various definitions which we make use of in the article. We refer to Diestel's book [25] for standard definitions from Graph Theory. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A graph G' is a *subgraph* of G if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. For a subset $V' \subseteq V(G)$, the subgraph $G' = G[V']$ of G is called the *subgraph induced by* V' if $E(G') = \{uv \in E(G) \mid u, v \in V'\}$. By $N_G(u)$, we denote the (open) neighborhood of u in graph G, that is, the set of all vertices adjacent to u and by $N_G[u] = N_G(u) \cup \{u\}$. Similarly, for a subset $D \subseteq V$, we define $N_G[D] = \cup_{v \in D} N_G[v]$ and $N_G(D) = N_G[D] \setminus D$. Given a set $S \subseteq V(G)$, we define $\partial_G(S)$ as the set of vertices in S that have a neighbor in $V(G) \setminus S$. We omit the subscripts when they are clear from the context. A subset of vertices D is called a *dominating set* of G if $N[D] = V(G)$. A subset of vertices D is called a *connected dominating set* if it is a dominating set and for every connected component C of G we have that $G[D \cap C]$ is connected. Throughout the article, given a graph G and vertex subsets Z and S, whenever we say that a subset Z *dominates all but (everything but)* S, then we mean that $V(G) \setminus S \subseteq N[Z]$. Observe that a vertex of S can also be dominated by the set Z.

We denote by K_h the complete graph on h vertices. Also for a given graph G and a vertex subset S, by $K[S]$ we mean a clique on the vertex set S. For an integer $r \geq 1$ and vertex subsets $P, Q \subseteq V(G)$, we say that a subset Q is r-*dominated* by P if for every $v \in Q$ there is $u \in P$ such that the distance between u and v is at most r. For $r = 1$, we simply say that Q is dominated by P. We denote by $N_G^r(P)$ the set of vertices r-dominated by P.

Throughout this article, we use $\mathbb{Z}$, $\mathbb{Z}^+$, and $\mathbb{Z}^-$ for the sets of integers, non-negative, and non-positive integers, respectively. Finally, we use $\mathbb{N}$ for the set of positive integers.

Minors and Contractions. Given an edge $e = xy$ of a graph G, the graph G/e is obtained from G by contracting the edge e, that is, the endpoints x and y are replaced by a new vertex v_{xy} that is adjacent to the old neighbors of x and y (except from x and y). A graph H obtained by a sequence of edge-contractions is said to be a *contraction* of G. We denote it by $H \leq_c G$. A graph H is a *minor* of a graph G if H is the contraction of some subgraph of G, and we denote it by $H \leq_m G$. We say that a graph G is H-minor-free when it does not contain H as a minor. We also say that a graph class $\mathcal{G}_H$ is H-minor-free (or excludes H as a minor) when all its members are H-minor-free. An *apex graph* is a graph obtained from a planar graph G by adding a vertex and making it adjacent to some of the vertices of G. A graph class $\mathcal{G}_H$ is *apex-minor-free* if $\mathcal{G}_H$ excludes a fixed apex graph H as a minor.

A *subdivision* of a graph H is obtained by replacing each edge of H by a non-trivial path. We say that H is a *topological minor* of G if some subgraph of G is isomorphic to a subdivision of H and denote it by $H \preceq_T G$. A graph G *excludes a graph* H *as a (topological) minor* if H is not a (topological) minor of G. For a graph H, by C_H, we denote all graphs that exclude H as topological minors.

Tree-Decompositions. A *tree-decomposition* of a graph G is a pair (M, Ψ), where M is a rooted tree and $\Psi : V(M) \to 2^{V(G)}$, such that

(1) $\bigcup_{t \in V(M)} \Psi(t) = V(G)$.
(2) For each edge $uv \in E(G)$, there is a $t \in V(M)$ such that both u and v belong to $\Psi(t)$.
(3) For each $v \in V(G)$, the nodes in the set $\{t \in V(M) \mid v \in \Psi(t)\}$ form a subtree of M.

If M is a path, then we call the pair (M, Ψ) as *path-decomposition.*
The following notations are the same as that in Reference [47]. Given a tree-decomposition of a graph G, we define mappings $\sigma, \gamma : V(M) \to 2^{V(G)}$ and $\kappa : E(M) \to 2^{V(G)}$. For all $t \in V(M)$,

$$\sigma(t) = \begin{cases} \emptyset & \text{if } t \text{ is the root of } M \\ \Psi(t) \cap \Psi(s) & \text{if } s \text{ is the parent of } t \text{ in } M \end{cases}$$

$$\gamma(t) = \bigcup_{u \text{ is a descendant of } t} \Psi(u).$$

For all $e = uv \in E(M)$, $\kappa(e) = \Psi(u) \cap \Psi(v)$.
For a subgraph M' of M by $\Psi(M')$ we denote $\cup_{t \in V(M')} \Psi(t)$.
Let (M, Ψ) be a tree-decomposition of a graph G. The *width* of (M, Ψ) is

$$\min\{|\Psi(t)| - 1 \mid t \in V(M)\},$$

and the *adhesion* of the tree-decomposition is

$$\max\{|\sigma(t)| \mid t \in V(M)\}.$$

We use $\mathbf{tw}(G)$ to denote the treewidth of the input graph. For every node $t \in V(M)$, the *torso* at t is the graph

$$\tau(t) := G[\Psi(t)] \cup E(K[\sigma(t)]) \cup \bigcup_{u \text{ child of } t} E(K[\sigma(u)]).$$

We take the graph induced by $\Psi(t)$, turn $\sigma(t)$ into a clique, and make vertices x, y adjacent if they appear together in the separator of some child u of t.

Parameterized Graph Problems. A parameterized graph problem Π is usually defined as a subset of $\Sigma^* \times \mathbb{Z}^+$ where, in each instance (x, k) of Π, x encodes a graph and k is the parameter (we denote by $\mathbb{Z}^+$ the set of all non-negative integers). In this article, we use an extension of this definition (also used by Bodlaender et al. [12]) that permits the parameter k to be negative with the additional constraint that either all pairs with non-positive values of the parameter are in Π or that no such pair is in Π. Formally, a parametrized problem Π is a subset of $\Sigma^* \times \mathbb{Z}$ where for all $(x_1, k_1), (x_2, k_2) \in \Sigma^* \times \mathbb{Z}$ with $k_1, k_2 < 0$ it holds that $(x_1, k_1) \in \Pi$ if and only if $(x_2, k_2) \in \Pi$. This extended definition encompasses the traditional one and is needed for technical reasons (see Section 3.2). In an instance of a parameterized problem (x, k), the integer k is called the parameter. Now we formally define the DS and CDS problems.

DS	**Parameter:** k
Input: An undirected graph G and a positive integer k.	
Question: Does there exists $D \subseteq V(G)$ of size at most k such that $N[D] = V(G)$?	

CDS	**Parameter:** k
Input: An undirected graph G and a positive integer k.	
Question: Does there exists $D \subseteq V(G)$ of size at most k such that $N[D] = V(G)$ and $G[D]$ is connected?	

Kernels and Protrusions. A central notion in parameterized complexity is *fixed parameter tractability*, which means, for a given instance (x, k), solvability in time $f(k) \cdot p(|x|)$, where f is an arbitrary function of k and p is a polynomial function in the input size. The notion of *kernelization* is formally defined as follows.

Definition 2.1. A *kernelization algorithm*, or simply a *kernel*, for a parameterized problem Π is an algorithm $\mathcal{A}$ that, given an instance (x, k) of Π, works in polynomial-time and returns an equivalent instance (x', k') of Π. Moreover, there exists a computable function $g(\cdot)$ such that whenever (x', k') is the output for an instance (x, k), then it holds that $|x'| + k' \leq g(k)$. If the upper bound $g(\cdot)$ is a polynomial (linear) function of the parameter, then we say that Π admits a *polynomial (linear) kernel*.

We often abuse the notation and call the output of a kernelization algorithm, the "reduced" equivalent instance, also a kernel.

Definition 2.2. Given a graph G, we say that a set $X \subseteq V(G)$ is an r*-protrusion* of G if $\mathbf{tw}(G[X]) \leq r$ and the number of vertices in X with a neighbor in $V(G) \setminus X$ is at most r.

2.1 Known Decomposition Theorems

We start with the definition of nearly embeddable graphs.

Definition 2.3 (h-nearly embeddable graphs). Let Σ be a surface with boundary cycles $C_1, \ldots, C_h$, that is, each cycle C_i is the border of a disc in Σ. A graph G is h*-nearly embeddable* in Σ if G has a subset X of size at most h, called *apices*, such that there are (possibly empty) subgraphs $G_0 = (V_0, E_0), \ldots, G_h = (V_h, E_h)$ of $G \setminus X$ such that

- $G \setminus X = G_0 \cup \cdots \cup G_h$,
- G_0 is embeddable in Σ, we fix an embedding of G_0,
- graphs $G_1, \ldots, G_h$ (called *vortices*) are pairwise disjoint,
- for $1 \leq i \leq h$, let $U_i := \{u_{i_1}, \ldots, u_{i_{m_i}}\} = V_0 \cap V_i$, G_i has a path decomposition (B_{ij}, Ψ_{ij}), $1 \leq j \leq m_i$, of width at most h such that
  - for $1 \leq i \leq h$ and for $1 \leq j \leq m_i$ we have $u_{i_j} \in B_{ij}$
  - for $1 \leq i \leq h$, we have $V_0 \cap C_i = \{u_{i_1}, \ldots, u_{i_{m_i}}\}$ and the points $u_{i_1}, \ldots, u_{i_{m_i}}$ appear on C_i in this order (either if we walk clockwise or anti-clockwise).

The decomposition theorem that we use extensively for our proofs is given in the next theorem.

Theorem 2.4 (References [32, 47, 58]). *For every graph H, there exists a constant h, depending only on the size of H, such that every graph G with $H \not\preceq_T G$, there is a tree-decomposition (M, Ψ) of adhesion at most h such that for all $t \in V(M)$, one of the following conditions is satisfied:*

(1) $\tau(t)$ is h-nearly embedded in a surface Σ in which H cannot be embedded.
(2) $\tau(t)$ has at most h vertices of degree larger than h.

Moreover, if G is an H-minor-free then nodes of second type do not exist. Furthermore, there is an algorithm that, given graphs G, H on n and $|V(H)|$ vertices, respectively, computes such a tree-decomposition in time $f(|V(H)|)n^{O(1)}$ for some computable function f and, moreover, computes an apex set Z_t of size at most h for every bag of the first type.

One of the main consequence of Theorem 2.4 we need for our purposes is that (in the case when G is H-minor-free) for every H there exist constants h and h' such that for every torso L of the decomposition from Theorem 2.4, there exists a set of vertices $A \subseteq V(L)$ of size at most h, called apices, such that the graph obtained from L after deleting the apices does not contain some apex graph H' of size h' as a minor. See, for example, Reference [46, Theorem 13].

Furthermore we can assume that in (M, Ψ), for any $x, y \in V(M)$, $\Psi(x) \nsubseteq \Psi(y)$. That is, no bag is contained in other. See Reference [36, Lemma 11.9] for the proof.

2.2 Known Approximation Algorithms

Recall that by C_H we denote the class of graphs that exclude a fixed graph H as a topological minor. In this subsection, we state known polynomial-time constant factor approximation algorithms for DS and CDS on C_H. It is well known that graphs in C_H has bounded degeneracy. The following is known about the approximation of DS.

Lemma 2.5 (Reference [33]). *Let H be a graph. Then there exists a constant $\eta(H)$ depending only on $|V(H)|$ such that Dominating Set admits a $\eta(H)$-factor approximation algorithm on C_H.*

For CDS we need the following proposition attributed to Reference [31].

Proposition 2.6. *Let G be a connected graph and let Q be a dominating set of G such that $G[Q]$ has at most ρ connected components. Then there exists a set $Z \subseteq V(G)$ of size at most $2 \cdot (\rho - 1)$ such that $Q \cup Z$ is a connected dominating set in G and, given Q, we can find such a set Z in polynomial time.*

Combining Lemma 2.5 and Proposition 2.6 we arrive at the following:

Lemma 2.7. *Let H be a graph and $\eta(H)$ the constant from Lemma 2.5. Then CDS admits a $3\eta(H)$-factor approximation algorithm on C_H.*

3 A NEW ALGORITHM FOR PROTRUSION REPLACEMENT

In the next section, we introduce the notion of a "generalized protrusion." Recall that a protrusion in a graph is a subgraph of constant treewidth that is separated from the rest of the graph by at most a constant number of vertices. In our variant of protrusions, instead of stipulating that the subgraph be of constant treewidth, we ask that it contains a constant number of vertices from a solution. In this section, we show that even if we have a generalized protrusion, then we can find a replacement for it efficiently. We first start with some relevant definitions and concepts.

3.1 Boundaried Graphs

Here we define the notion of *boundaried graphs* and various operations on them.

Definition 3.1. (Boundaried Graphs). A boundaried graph is a graph G with a set $B \subseteq V(G)$ of distinguished vertices and an injective labelling λ from B to the set $\mathbb{Z}^+$. The set B is called the *boundary* of G, and the vertices in B are called *boundary vertices* or *terminals*. Given a boundaried graph G, we denote its boundary by $\delta(G)$, we denote its labelling by λ_G, and we define its *label set* by $\Lambda(G) = \{\lambda_G(v) \mid v \in \delta(G)\}$. Given a finite set $I \subseteq \mathbb{Z}^+$, we define $\mathcal{F}_I$ to denote the class of all boundaried graphs whose label set is I. We also denote by $\mathcal{F}$ the class of all boundaried graphs. Finally, we say that a boundaried graph is a t-*boundaried* graph if $\Lambda(G) \subseteq \{1, \ldots, t\}$.

Definition 3.2. (Gluing by $\oplus$). Let G_1 and G_2 be two boundaried graphs. We denote by $G_1 \oplus G_2$ the graph (not boundaried) obtained by taking the disjoint union of G_1 and G_2 and identifying equally-labeled vertices of the boundaries of G_1 and G_2. In $G_1 \oplus G_2$, there is an edge between two vertices if there is an edge between them in G_1 or in G_2 or both.

We remark that if G_1 has a label that is not present in G_2, or vice versa, then in $G_1 \oplus G_2$ we just forget that label.

Definition 3.3. (Gluing by $\oplus_\delta$). The *boundaried gluing operation* $\oplus_\delta$ is similar to the normal gluing operation but results in a boundaried graph rather than a graph. Specifically, $G_1 \oplus_\delta G_2$ results in a boundaried graph where the graph is $G = G_1 \oplus G_2$ and a vertex is in the boundary of G if it was in the boundary of G_1 or of G_2. Vertices in the boundary of G keep their label from G_1 or G_2.

Let $\mathcal{G}$ be a class of (not boundaried) graphs. By slightly abusing notation, we say that a boundaried graph *belongs to a graph class* $\mathcal{G}$ if the underlying graph belongs to $\mathcal{G}$.

Definition 3.4. (Replacement). Let G be a t-boundaried graph containing a set $X \subseteq V(G)$ such that $\partial_G(X) = \delta(G)$. Let G_1 be a t-boundaried graph. The result of *replacing* X *with* G_1 is the graph $G^\star \oplus G_1$, where $G^\star = G \setminus (X \setminus \partial(X))$ is treated as a t-boundaried graph with $\delta(G^\star) = \delta(G)$.

3.2 Finite Integer Index

Definition 3.5. (Canonical equivalence on boundaried graphs). Let Π be a parameterized graph problem whose instances are pairs of the form (G, k). Given two boundaried graphs $G_1, G_2 \in \mathcal{F}$, we say that $G_1 \equiv_\Pi G_2$ if $\Lambda(G_1) = \Lambda(G_2)$ and there exists a *transposition constant* $c \in \mathbb{Z}$ such that

$$\forall (F, k) \in \mathcal{F} \times \mathbb{Z} \quad (G_1 \oplus F, k) \in \Pi \Leftrightarrow (G_2 \oplus F, k + c) \in \Pi.$$

Here, c is a function of the two graphs G_1 and G_2.

Note that the relation $\equiv_\Pi$ is an equivalence relation. Observe that c could be negative in the preceding definition. This is the reason we allow the parameter in parameterized problem instances to take negative values.

Next, we define a notion of "transposition-minimality" for the members of each equivalence class of $\equiv_\Pi$.

Definition 3.6. (Progressive representatives). Let Π be a parameterized graph problem whose instances are pairs of the form (G, k), and let C be some equivalence class of $\equiv_\Pi$. We say that $J \in C$ is a *progressive representative* of C if for every $H \in C$ there exists $c \in \mathbb{Z}^-$, such that

$$\forall (F, k) \in \mathcal{F} \times \mathbb{Z} \quad (H \oplus F, k) \in \Pi \Leftrightarrow (J \oplus F, k + c) \in \Pi. \tag{1}$$

The following lemma guarantees the existence of a progressive representative for each equivalence class of $\equiv_\Pi$.

Lemma 3.7 ([12]). *Let Π be a parameterized graph problem whose instances are pairs of the form (G, k). Then each equivalence class of $\equiv_\Pi$ has a progressive representative.*

Notice that two boundaried graphs with different label sets belong to different equivalence classes of $\equiv_\Pi$. Hence, for every equivalence class C of $\equiv_\Pi$, there exists some finite set $I \subseteq \mathbb{Z}^+$ such that $C \subseteq \mathcal{F}_I$. We are now in position to give the following definition:

Definition 3.8. (Finite Integer Index). A parameterized graph problem Π whose instances are pairs of the form (G, k) has *Finite Integer Index* (or is *FII*) if and only if, for every finite $I \subseteq \mathbb{Z}^+$, the number of equivalence classes of $\equiv_\Pi$ that are subsets of $\mathcal{F}_I$ is finite. For each $I \subseteq \mathbb{Z}^+$, we define $\mathcal{S}_I$ to be a set containing exactly one progressive representative of each equivalence class of $\equiv_\Pi$ that is a subset of $\mathcal{F}_I$. We also define $\mathcal{S}_{\subseteq I} = \bigcup_{I' \subseteq I} \mathcal{S}_{I'}$.

3.3 Replacement Lemma

We first define a notion of monotonicity for parameterized problems.

Definition 3.9. We say that a parameterized graph problem Π is *positive monotone* if, for every graph G, there exists a unique $\ell \in \mathbb{N}$ such that for all $\ell' \in \mathbb{N}$ and $\ell' \geq \ell$, $(G, \ell') \in \Pi$ and for all $\ell' \in \mathbb{N}$ and $\ell' < \ell$, $(G, \ell') \notin \Pi$. A parameterized graph problem Π is *negative monotone* if for every graph G there exists a unique $\ell \in \mathbb{N}$ such that for all $\ell' \in \mathbb{N}$ and $\ell' \geq \ell$, $(G, \ell') \notin \Pi$ and for all $\ell' \in \mathbb{N}$ and $\ell' < \ell$, $(G, \ell') \in \Pi$. Π is monotone if it is either positive monotone or negative monotone. We denote the integer ℓ by $\textsc{Threshold}(G, \Pi)$ (in short $\textsc{Thr}(G, \Pi)$).

We first give an intuition for the next definition. We are considering monotone functions and thus for every graph G there is an integer k where the answer flips. However, for our purpose, we need a corresponding notion for boundaried graphs. If we think of the representatives as some "small perturbation," then it is the max threshold over all small perturbations ("adding a representative = small perturbation"). This leads to the following definition:

Definition 3.10. Let Π be a monotone parameterized graph problem that has FII. Let $\mathcal{S}_t$ be a set containing exactly one progressive representative of each equivalence class of $\equiv_\Pi$ that is a subset of $\mathcal{F}_I$, where $I = \{1, \ldots, t\}$. For a t-boundaried graph G, we define

$$\iota(G) = \max_{G' \in \mathcal{S}_t} \textsc{Thr}(G \oplus G', \Pi).$$

The next lemma says the following: Suppose we are dealing with some FII problem and we are given a boundaried graph with constant size boundary. We know it has some constant size representative, and we want to find this representative. Now, in general, finding a representative for a boundaried graph is more difficult than solving the corresponding problem. The next lemma says basically that if "OPT" of a boundaried graph is low, then we can efficiently find its representative. Here by "OPT" we mean $\iota(G)$, which is a robust version of the threshold function (under adding a representative), and by "efficiently," we mean as efficiently as solving the problem on normal (unboundaried) graphs if we know that "OPT" is low. Specifically, the main result of this section is as follows:

Lemma 3.11. *Let Π be a monotone parameterized graph problem that has FII. Furthermore, let $\mathcal{A}$ be an algorithm for Π that, given a pair (G, k), decides whether it is in Π in time $f(|V(G)|, k)$. Then for every $t \in \mathbb{N}$, there exists a $\xi_t \in \mathbb{Z}^+$ (depending on Π and t) and an algorithm that, given a t-boundaried graph G with $|V(G)| > \xi_t$, outputs, in $\mathcal{O}(\iota(G)(f(|V(G)| + \xi_t, \iota(G)))$ steps, a t-boundaried graph G^* such that $G \equiv_\Pi G^*$ and $|V(G^*)| < \xi_t$. Moreover, we can compute the translation constant c from G to G^* in the same time.*

Proof. We give prove the claim for positive monotone problems Π; the proof for negative monotone problems is identical. Let $\mathcal{S}_t$ be a set containing exactly one progressive representative of each equivalence class of $\equiv_\Pi$ that is a subset of $\mathcal{F}_I$, where $I = \{1, \ldots, t\}$, and let $\xi_t = \max_{Y \in \mathcal{S}_t} |V(Y)|$. The set $\mathcal{S}_t$ is hardwired in the description of the algorithm. Let $Y_1, \ldots, Y_\rho$ be the set of progressive representatives in $\mathcal{S}_t$. Let $\mathcal{F}_t = \mathcal{F}_I$. Our objective is to find a representative Y_ℓ for G such that

$$\forall (F, k) \in \mathcal{F}_t \times \mathbb{Z} \quad (G \oplus F, k) \in \Pi \Leftrightarrow (Y_\ell \oplus F, k - \vartheta(X, Y_\ell)) \in \Pi. \tag{2}$$

Here, $\vartheta(X, Y_\ell)$ is a constant that depends on G and Y_ℓ. Towards this we define the following matrix for the set of representatives. Let

$$A[Y_i, Y_j] = \textsc{Thr}(Y_i \oplus Y_j, \Pi).$$

The matrix A has constant size and is also hardwired in the description of the algorithm.

Now, given G, we find its representative as follows:

- Compute the following row vector $\mathcal{X} = [\textsc{Thr}(G \oplus Y_1, \Pi), \ldots, \textsc{Thr}(G \oplus Y_\rho, \Pi))]$. For each Y_i, we decide whether $(G \oplus Y_i, k) \in \Pi$ using the assumed algorithm for deciding Π, letting k increase from 1 until the first time $(G \oplus Y_i, k) \in \Pi$. Since Π is positive monotone, this will happen for some $k \leq \iota(G)$. Thus the total time to compute the vector $\mathcal{X}$ is $O(\iota(G)(f(|V(G)| + \xi_t, \iota(G)))$.
- Find a translate row in the matrix $A(\Pi)$. That is, find an integer n_o and a representative Y_ℓ such that

$$\begin{aligned}&[\textsc{Thr}(G \oplus Y_1, \Pi), \textsc{Thr}(G \oplus Y_2, \Pi), \ldots, \textsc{Thr}(G \oplus Y_\rho, \Pi)]\\ &= [\textsc{Thr}(Y_\ell \oplus Y_1, \Pi) + n_0, \textsc{Thr}(Y_\ell \oplus Y_2, \Pi) + n_0, \ldots, \textsc{Thr}(Y_\ell \oplus Y_\rho, \Pi) + n_0].\end{aligned}$$

  Such a row must exist, since $\mathcal{S}_t$ is a set of representatives for Π; the representative Y_ℓ for the equivalence class to which G belongs satisfies the condition.
- Set Y_ℓ to be G^* and the translation constant to be $-n_0$.

From here, it easily follows that $G \equiv_\Pi G^*$. This completes the proof. □

We remark that the algorithm whose existence is guaranteed by the Lemma 3.11 assumes that the set $\mathcal{S}_t$ of representatives are hardwired in the algorithm. In its full generality, we currently do not know of a procedure that, for problems having FII, outputs such a representative set. The application of Lemma 3.11 makes our kernelization algorithm non-constructive.

4 GENERALIZED PROTRUSIONS AND SLICE-DECOMPOSITION

In this section, our objective is to show that in polynomial time we can partition the graph G into parts that satisfy certain properties. To obtain our decomposition, we need to use a more general notion of protrusion. Recall that a protrusion in a graph is a subgraph of constant treewidth that is separated from the rest of the graph by at most a constant number of vertices. In our variant of protrusions, instead of stipulating that the subgraph be of constant *treewidth*, we ask that it contains a *constant number of vertices from a solution*. More precisely, we need the following kind of protrusions:

Definition 4.1 (r-DS-protrusion). Given a graph G, we say that a set $X \subseteq V(G)$ is an *r-DS-protrusion* of G if the number of vertices in X with a neighbor in $V(G) \setminus X$ is at most r and there exists a subset $S \subseteq X$ of size at most r such that S is a dominating set of $G[X]$.

The notion of a r-DS-protrusion X differs from a protrusion in the following way. In a protrusion $\mathbf{tw}(X)$ is at most r, while in a r-DS-protrusion we require that the dominating set of the graph induced by X is small. In the case of a r-DS-protrusion, the treewidth could be unbounded. One can similarly define the notion of a r-Π-protrusion for other graph problems Π. Next we define a r-CDS-protrusion.

Definition 4.2 (r-CDS-protrusion). Given a graph G, we say that a set $X \subseteq V(G)$ is an *r-CDS-protrusion* of G if the number of vertices in X with a neighbor in $V(G) \setminus X$ is at most r and there exists a subset $S \subseteq X$ of size at most r such that for every connected component C of $G[X]$ we have that $S \cap C$ is connected and is a dominating set for C.

A natural question is what can we do if we get a large r-DS-protrusion (or r-CDS-protrusion). The next lemma shows that in that case we can replace it with an equivalent small graph. We will also need the following. Let $\mathcal{G}$ be a graph class. Given a parameterized graph problem Π and a graph class $\mathcal{G}$, we denote by $\Pi \Cap \mathcal{G}$ the problem obtained by removing from Π all instances that encode graphs that do not belong to $\mathcal{G}$. Our result is as follows.

Lemma 4.3. *Let H be a fixed graph. For every $t \in \mathbb{Z}^+$, there exist a $\xi_t \in \mathbb{Z}^+$ (depending on DS (CDS), t and H) and an algorithm $\mathcal{A}$ such that, given a t-DS-protrusion X (t-CDS-protrusion) with boundary $\partial(X)$, $|V(X)| > \xi_t$, and $H \not\preceq_T X$, $\mathcal{A}$ outputs in $O(|V(X)|)$ time ($|V(X)|^{O(1)}$ time), a t-boundaried graph X' such that $H \not\preceq_T X'$ ($H \not\preceq_m X'$) and $X \equiv_{\mathrm{DS}} X'$ ($X \equiv_{\mathrm{CDS}} X'$) and $|V(X')| \leq \xi_t$. Moreover, at the same time, we can also find the translation constant c from X to X'.*

Proof. Let $\mathcal{G}$ be the class of graphs that excludes H as a topological minor. For every $t \in \mathbb{Z}^+$, let ξ_t be the constant as defined in Lemma 3.11. It is known that DS$\Cap\mathcal{G}$ (CDS$\Cap\mathcal{G}$) is FII\cal [12] and monotone (see Reference [12, Lemmas 7.3 and 7.4]). Furthermore, DS and CDS can be solved in time $O((hk)^{hk}n)$ [5, Theorem 4] and $O(k^{O(h^2)k}n^{O(1)})$ [45, Theorem 1], respectively. Here, $h = |V(H)|$ and k is the parameter in the definitions of DS and CDS. We use these algorithms in Lemma 3.11 with the parameter value being r. That is, $k := r$. Thus, if $|V(X)| > \xi_t$, then by Lemma 3.11 in time $O(|V(X)|)$ ($|V(X)|^{O(1)}$), we can obtain a t-boundaried graph X' such that $X \equiv_{\mathrm{DS}} X'$ ($X \equiv_{\mathrm{CDS}} X'$), $|V(X')| < \xi_t$, and $H \not\preceq_T X'$. The last assertion that $H \not\preceq_T X'$ follows from the fact that DS$\Cap\mathcal{G}$ is FII, and thus all the graphs in the set of representatives with respect to $\equiv_{\mathrm{DS}}$ belong to $\mathcal{G}$. Moreover, at the same time, we can also find the translation constant c from X to X', as done in Lemma 3.11.

Let $\mathcal{G}^\star$ be the class of graphs that excludes a fixed graph H as a minor. It is known that DS$\Cap\mathcal{G}^\star$ (CDS$\Cap\mathcal{G}^\star$) is FII\cal [12] and monotone. Thus, as in the case of $\mathcal{G}$, we can obtain a t-boundaried graph X' such that $X \equiv_{\mathrm{DS}} X'$ ($X \equiv_{\mathrm{CDS}} X'$), $|V(X')| < \xi_t$, and $H \not\preceq_m X'$. □

Throughout this section we work on a graph G that excludes a fixed graph H as a topological minor. Here, h will denote $|V(H)|$.
Furthermore, we assume that we have a (connected) dominating set D such that the size of D is at most $\eta(H)$-factor away ($3\eta(H)$-factor away) from the size of an optimal (connected) dominating set of G, obtained by applying Lemma 2.5 (Lemma 2.7) on the input graph G.

Let (M, Ψ) be a tree-decomposition of a graph G. For a subtree M_i of M, we define $\mathcal{E}(M_i)$ as the set of edges in M such that it has exactly one endpoint in $V(M_i)$. Furthermore, we define $R_i^+ = \Psi(M_i)$ and

$$\tau(M_i) := G[R_i^+] \cup \bigcup\nolimits_{e \in \mathcal{E}(M_i)} E(K[\kappa(e)]).$$

To put it simply, R_i^+ denotes the union of bags corresponding to the nodes in M_i and $\tau(M_i)$ is the graph induced on R_i^+ with "external adhesions" being torsoed.

Our main objective in this section is to obtain the following (α, β)-*slice decomposition* for $\alpha = \beta = O(k)$.

Definition 4.4 ((α, β)-slice decomposition). Let H be a fixed graph and let G be a graph with $H \not\preceq_T G$. Let (M, Ψ) be the tree-decomposition given by Theorem 2.4. An (α, β)-*slice decomposition* of a graph G is a collection $\mathcal{P}$ of pairwise vertex disjoint subtrees $\{M_1, \ldots, M_\alpha\}$ of M such that the following hold:

- $\bigcup_{1 \leq i \leq \alpha} \Psi(M_i) = \bigcup_{1 \leq i \leq \alpha} (\bigcup_{t \in V(M_i)} \Psi(t)) = V(G)$.
- There exists a graph H^* whose size only depends on h, such that each $\tau(M_i)$ is either H^*-minor-free or has at most h vertices of degree at least h.
- $$\sum_{i=1}^{\alpha} \left(\sum_{e \in \mathcal{E}(M_i)} |\kappa(e)| \right) \leq \beta.$$

We refer to the sets R_i^+, $i \in \{1, \ldots, \alpha\}$, as the *slices* of $\mathcal{P}$.

Essentially, the slice decomposition allows us to partition the input graph G into subgraphs $C_0, C_1, \ldots, C_\ell$, such that $|C_0| = O(k)$; for every $i \geq 1$, the neighbourhood $N(C_i) \subseteq C_0$, and $\sum_{1 \leq i \leq \ell} |N(C_i)| = O(k)$. To see this consider an instance (G, k) of DS, where G excludes a fixed graph H as a topological minor. Now obtain an (α, β)-slice decomposition for $\alpha = \beta = O(k)$ for G. We take

$$C_0 = \bigcup_{i=1}^{\alpha} \left(\cup_{e \in \mathcal{E}(M_i)} \kappa(e) \right),$$

and $C_i = \Psi(M_i) \setminus C_0$. One can easily verify that this partition of $V(G)$ satisfies the stated properties. This is the decomposition we were talking about in the Introduction.

Now we give a definitions that is useful in our procedure to find the slice decomposition.

Definition 4.5. Let (M, Ψ) be the tree-decomposition of a graph G given by Theorem 2.4. For a subset $Q \subseteq V(G)$ and a subtree M' of M, we define $\mu(M', Q) = |\Psi(M') \cap Q|$.

Let (M, Ψ) be the tree-decomposition of a graph G given by Theorem 2.4. If we delete an edge $e = uv \in E(M)$ from the tree M, then we get two trees. We call the *trees M_u and M_v based on whether they contain u or v.*

Definition 4.6. Let (M, Ψ) be the tree-decomposition of a graph G given by Theorem 2.4 and D be the assumed dominating (connected) set of G. We call a tree edge $e = uv \in E(M)$ *heavy* if $\mu(M_u, D) \geq h + 1$ and $\mu(M_v, D) \geq h + 1$. We use $\mathcal{F}$ to denote the set of heavy edges.

Our main lemma in this section shows that in polynomial time we can find an $(O(k), O(k))$-slice decomposition or a large r-DS-protrusion (or r-CDS-protrusion) or a large protrusion. In the latter cases, we can apply either Lemma 4.3 or a similar lemma developed in Reference [12, Lemma 7] for protrusions and reduce the graph. Before we prove the main result of this section, we prove some combinatorial properties of the set $\mathcal{F}$. Given $\mathcal{F}$, by *"a subgraph of M formed by the edges in $\mathcal{F}$,"* we mean a subgraph of M whose vertex set consists of the end points of edges in $\mathcal{F}$ and the edge set is $\mathcal{F}$.

Lemma 4.7. *Let M^* be the subgraph of M formed by the edges in $\mathcal{F}$. Then M^* is a subtree of M.*

Proof. Clearly, M^* is a forest, as it is a subgraph of M. To complete the proof, we need to show that it is connected. We prove this using contradiction. Suppose M^* is a forest and M_i^* and M_j^*, $i \neq j$, are two maximal subtrees in M^*. Then we know that there exists a path P in M such that the first and the last edges are heavy and the path P contains a light edge. Furthermore, we can assume that the first edge, say, $u_i v_i$, belongs to M_i^* and the last edge, say, $u_j v_j$, belongs to M_j^*. Let a light edge on the path be xy. Now, when we delete the edge xy from M, we get two trees, M_x and M_y. We know that either $M_i^* \subseteq M_x$ and $M_j^* \subseteq M_y$ or vice versa. Suppose $M_i^* \subseteq M_x$ and $M_j^* \subseteq M_y$. Since M_i^* contains the heavy edge $u_i v_i$, we have that $\mu(M_x, D) \geq h + 1$. Similarly, we can show that $\mu(M_y, D) \geq h + 1$. This shows that xy is a heavy edge, contradicting that xy is light. One can similarly argue that xy is a heavy edge when $M_i^* \subseteq M_y$ and $M_j^* \subseteq M_x$. This contradicts our assumption that M^* is not a subtree of M. This completes the proof of the lemma. □

For our next proof, we first give some well-known observations about trees. Given a tree T, we call a node a *leaf*, *link*, or *branch* if its degree in T is 1, 2, or ≥3, respectively. Let $S_{\geq 3}(T)$ be the set of branch nodes, $S_2(T)$ be the set of link nodes, and $L(T)$ be the set of leaves in the tree T. Let $\mathscr{P}_2(T)$ be the set of maximal paths consisting entirely of link nodes.

Fact 1. $|S_{\geq 3}(T)| \leq |L(T)| - 1$.

Fact 2. $|\mathscr{P}_2(T)| \leq 2|L(T)| - 1$.

Proof. Root the tree at an arbitrary node of degree at least 3. If there is no node of degree 3 or more in T, then we know that T is a path and the assertion follows. Consider $T[S_2]$, which is the disjoint union of paths $P \in \mathscr{P}_2(T)$. With every path $P \in \mathscr{P}_2(T)$, we associate the unique child in T of the last node of this path (furtherest from the root) in T. Observe that this association is injective and the associated node is either a leaf or a branch node. Hence

$$|\mathscr{P}_2(T)| \leq |L(T)| + |S_{\geq 3}(T)| \leq 2|L(T)| - 1$$

follows from Fact 1. □

Lemma 4.8. *Let M^* be the subgraph formed by the edges in $\mathcal{F}$. If (G,k) is a yes instance of DS (CDS), then (a) $|L(M^*)| \leq \eta(H)k$; (b) $|S_{\geq 3}(M^*)| \leq \eta(H)k - 1$; and (c) $|\mathscr{P}_2(M^*)| \leq 2\eta(H)k - 1$. Here $\eta(H)$ is the factor of approximation in Lemma 2.5 (Lemma 2.7).*

Proof. Root the tree at an arbitrary node r of degree at least 3 in M^*. If there is no node of degree 3 or more in M^*, then we know that M^* is a path, and the proof follows. We call a pair of nodes u and v *siblings* if they do not belong to the same path from the root r in M^*. Observe that all the leaves of M^* are siblings.

Let D be an approximate solution to DS (CDS) returned by applying Lemma 2.5 (Lemma 2.7) on G. Since (G,k) is a yes instance, we have that $|D| \leq \eta(H)k$. Let $w_1, \ldots, w_\ell$ be the leaves of M^* and let $e_1, \ldots, e_\ell$ be the edges in M^* incident with $w_1, \ldots, w_\ell$, respectively. To prove our first statement, we will show that for every i, we have a vertex $q_i \in D$ such that $q_i \in \gamma(w_i)$ and for all $j \neq i$, $q_i \notin \gamma(w_j)$. This will establish an injection from the set of leaves to the dominating set D, and thus the bound will follow. Towards this, observe that since the edge e_i is heavy, we have that $|\gamma(w_i) \cap D| \geq h + 1$. Furthermore, for every pair of vertices $w_i, w_j \in L(M^*)$, $w_i \neq w_j$, we have that $|\gamma(w_i) \cap \gamma(w_j)| \leq h$. The last assertion follows from the fact that for a pair of siblings w_i and w_j the only vertices that can be in the intersection of $\gamma(w_i)$ and $\gamma(w_j)$ must belong to both $\sigma(w_i)$ and $\sigma(w_j)$. However, the size of any $\sigma(w_i)$ is upper bounded by h. This, together with the fact that $|\gamma(w_i) \cap D| \geq h + 1$ implies that for every i, we have a vertex $q_i \in D$ such that $q_i \in \gamma(w_i)$ and for all $j \neq i$, $q_i \notin \gamma(w_j)$. This implies that $|L(M^*)| \leq |D|$. However, since (G,k) is a yes instance to DS, we have that $|D| \leq \eta(H)k$. This completes the proof of part (a) of the lemma. Proofs for part (b) and part (c) of the lemma follow from Facts 1 and 2. □

Before we prove our next lemma we show a lemma about dominating sets of subgraphs of G.

Lemma 4.9. *Let H be a fixed graph and let G be a graph with $H \not\preceq_T G$. Let (M, Ψ) be the tree-decomposition of G given by Theorem 2.4 and let D be a dominating set of G. If M' is a subtree of M, then*

$$(D \cap \Psi(M')) \bigcup_{e \in \mathcal{E}(M')} \kappa(e)$$

is a dominating set for $G[\Psi(M')]$.

Proof. The proof follows from the fact that $D \cap \Psi(M')$ dominates all the vertices in $\Psi(M')$ except possibly the ones that have neighbors in $V(G) \setminus (\cup_{e \in \mathcal{E}(M')} \kappa(e))$. Thus,

$$(D \cap \Psi(M')) \bigcup_{e \in \mathcal{E}(M')} \kappa(e)$$

is a dominating set for $G[\Psi(M')]$. □

Let $P_1, \ldots, P_\ell$ be the paths in $\mathscr{P}_2(M^*)$. We use s_i and t_i to denote the first and the last vertices, respectively, of the path P_i. Since P_i is a path consisting of link vertices, we have that s_i and t_i have unique neighbors s_i^* and t_i^*, respectively, in M^*. Observe that since M^* is a subtree of M, then we have that for every i, P_i is also a path in M. If we delete the edges $s_i^*s_i$ and $t_i^*t_i$ from the tree M, then there is a subtree of M that contains the path P_i; we call this subtree $M(P_i)$. For any two vertices a and b on the path P_i, we use $P_i(a,b)$ to denote the subpath between a and b in P_i. Furthermore, for any subpath $P_i(a,b)$, if we delete the edges incident to a and b on P_i and not present in $P_i(a,b)$ from the tree M, then there is a subtree of M that contains the path $P_i(a,b)$; we call this subtree $M(P_i(a,b))$.

Now we recall the definition of ξ_t. Let Π be a monotone parameterized graph problem that is FII. Then, for every $t \in \mathbb{N}$, there exists a $\xi_t \in \mathbb{Z}^+$ (depending on Π and t), such that, given a t-boundaried graph G with $|V(G)| > \xi_t$, there exists a t-boundaried graph G^* such that $G \equiv_\Pi G^*$ and $|V(G^*)| < \xi_t$. In the next lemma, we show that if any of the paths is "too long," then using a simple application of pigeonhole principle, we can get a $2h$-DS-protrusion. We use $|P_i|$ to denote the number of vertices in the path P_i.

Lemma 4.10. *Let (G,k) be an instance of DS (CDS) and let $P_1,\ldots,P_\ell$ be the paths in $\mathscr{P}_2(M^*)$. Further, let D be a dominating set of G. Then, for some path P_i, $i \in \{1,\ldots,\ell\}$, if $|P_i| > \xi_{2h}(2(2h+k_i)+1)$, then G contains a $2h$-DS-protrusion ($2h$-CDS-protrusion) of size at least ξ_{2h}. Here, $k_i = |D \cap \Psi(M(P_i))|$. Furthermore, we can find such a $2h$-DS-protrusion ($2h$-CDS-protrusion) in polynomial time.*

Proof. Let P_i be the the path such that $|P_i| > 2\xi_{2h}(|D \cap \Psi(M(P_i))|$. Let $P_i := s_i = a_1^i \cdots a_\ell^i = t_i$. For every vertex

$$w \in (D \cap \Psi(M(P_i))) \cup \kappa(s_is_i^*) \cup \kappa(t_it_i^*),$$

we mark two vertices of the path P_i. We mark the first and the last vertices on P_i, say, a^i_{wfirst} and a^i_{wlast}, such that $w \in \Psi(a^i_{\text{wfirst}})$ and $w \in \Psi(a^i_{\text{wlast}})$. That is, $w \in \Psi(a^i_{\text{wfirst}})$ and $w \in \Psi(a^i_{\text{wlast}})$ and for all $z <$ wfirst or $z >$ wlast we have that $w \notin \Psi(a^i_z)$. This way, we will only mark at most $2(2h + |D \cap \Psi(M(P_i))|) = 2(2h+k_i)$ vertices of the path P_i. However, the path is longer than $2\xi_{2h}(2h+k_i)$, and thus, by the pigeonhole principle, we have that there exists a subpath of P_i, say, $P_i(a_x^i, a_y^i)$, such that no vertex of this subpath is marked and $|P_i(a_x^i,a_y^i)| > \xi_{2h}$. Let $W = \Psi(M(P_i(a_x^i,a_y^i)))$. Let a^* and b^* be the neighbors of a_x^i and a_y^i, respectively, that are not present on $P_i(a_x^i,a_y^i)$. Clearly, the only vertices in W that have neighbors in $V(G) \setminus W$ belong to $\kappa(a_x^ia^*) \cup \kappa(a_y^ib^*)$. Thus the vertices in W that have neighbors in $V(G) \setminus W$ is upper bounded by $2h$. Furthermore, since no vertex on the path $P_i(a_x^i,a_y^i)$ is marked, we have that all the vertices in D belonging to W also belong to $\kappa(a_x^ia^*) \cup \kappa(a_y^ib^*)$. Then, by Lemma 4.9, we have that $\kappa(a_x^ia^*) \cup \kappa(a_y^ib^*)$ dominates all the vertices in W. Furthermore, in (M,Ψ), no bag is contained in another and thus $|W| > \xi_{2h}$ (see discussion after Theorem 2.4). This shows that W is a $2h$-DS-protrusion of the desired size. □

The final result of this section is the following decomposition lemma:

Lemma 4.11. *Let H be a fixed graph and C_H be the class of graphs that excluds a fixed graph H as a topological minor. Then there exist two constants δ_1 and δ_2 (depending on DS (CDS)) such that given a yes instance (G,k) of DS (CDS), in polynomial time, we can find*

- *a $(\delta_1 k, \delta_2 k)$-slice decomposition,*
- *a $2h$-DS-protrusion (or $2h$-CDS-protrusion) of size more than ξ_{2h}, or*
- *an h'-protrusion of size more than $\xi_{h'}$ where h' depends only on h.*

Proof. Let (G,k) be a yes instance of DS (CDS). This implies that the size of the (connected) dominating set D returned by Lemma 2.5 (Lemma 2.7) is at most $\eta(H)k$. Let M^* be the subtree of M formed by heavy edges. By Lemma 4.8, we know that

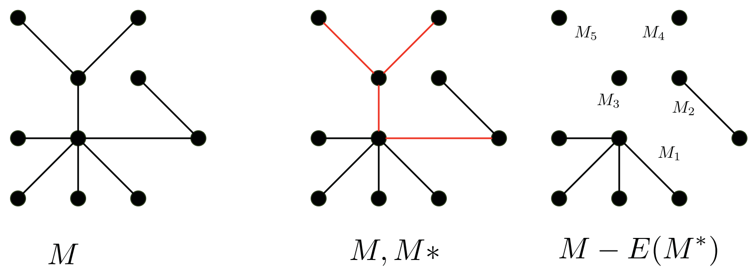

Fig. 2. An illustration of the decomposition. The heavy edges are shown in red.

(a) $|L(M^*)| \leq \eta(H)k$,
(b) $|S_{\geq 3}(M^*)| \leq \eta(H)k - 1$, and
(c) $|\mathscr{P}_2(M^*)| \leq 2\eta(H)k - 1$.

Recall that for every path $P_i \in \mathscr{P}_2(M^*)$, we defined $k_i = |D \cap \Psi(M(P_i))|$. If for any path $P_i \in \mathscr{P}_2(M^*)$ we have that $|P_i| > \xi_{2h}2(2h + k_i)$, then by Lemma 4.10 G contains a $2h$-DS-protrusion of size at least ξ_{2h}, and we can find this protrusion in polynomial time. Thus we assume that for all paths $P_i \in \mathscr{P}_2(M^*)$ we have that $|P_i| \leq \xi_{2h}(2(2h + k_i) + 1)$.

Let k_i^* denote the number of vertices in $D \cap \Psi(M(P_i))$ that are not present in any other $D \cap \Psi(M(P_j))$ for $i \neq j$. Furthermore, for all $i \neq j$, we have that

$$|\Psi(M(P_i)) \cap D \cap \Psi(M(P_j)| \leq h.$$

The last assertion is based on the following arguments. The sets $\Psi(M(P_i))$ and $\Psi(M(P_j))$ can be separated by a separator of size at most h and the vertices of D that appear in both sets are present in this separator. Observe that $k_i \leq 2h + k_i^*$. This implies that

$$\begin{aligned}
|V(M^*)| &= |L(M^*)| + |S_{\geq 3}| + |S_2| \\
&\leq \eta(H)k + \eta(H)k - 1 + \sum_{P_j \in \mathscr{P}_2(M^*)} (4h + 2k_j + 2)\xi_{2h} \\
&= 2\eta(H)k - 1 + (4h + 2)|\mathscr{P}_2(M^*)|\xi_{2h} + \sum_{P_j \in \mathscr{P}_2(M^*)} 2(2h + k_j^*)\xi_{2h} \\
&\leq 2\eta(H)k - 1 + (8h + 2)|\mathscr{P}_2(M^*)|\xi_{2h} + 2|D|\xi_{2h} \\
&\leq (2 + (16h + 4)\xi_{2h} + 2\xi_{2h})\eta(H)k.
\end{aligned}$$

Let $\Gamma = (2 + (16h + 4)\xi_{2h} + 2\xi_{2h})\eta(H)k$. This implies that the number of heavy edges is upper bounded by $|\mathcal{F}| \leq \Gamma - 1$. Let $M_1, \ldots, M_\alpha$ be the subtrees of M obtained by deleting all the edges in M^*, that is, by deleting all the edges in $\mathcal{F}$, see Figure 2 for an illustration. Note that

$$\alpha \leq \Gamma = (2 + (16h + 4)\xi_{2h} + 2\xi_{2h})\eta(H)k.$$

We now argue that either the collection $M_1, \ldots, M_\alpha$ forms a $(\delta_1 k, \delta_2 k)$-slice decomposition of G or we have found a $2h$-protrusion or a $2h$-DS-protrusion of size more than ξ_{2h} in G.

First, we show that

$$\sum_{i=1}^{\alpha} \left(\sum_{e \in \mathcal{E}(M_i)} |\kappa(e)| \right) = O(k).$$

Note that, by construction, each $e \in \mathcal{E}(M_i)$ is a heavy edge. Now observe that each e belongs to at most two distinct edge sets $\mathcal{E}(M_i)$, we have that

$$\sum_{i=1}^{\alpha} \sum_{e \in \mathcal{E}(M_i)} |\kappa(e)| \leq \left(2 \sum_{e \in E(M^*) = \mathcal{F}} |\kappa(e)| \right) \leq 2h|\mathcal{F}| \leq 2h\Gamma.$$

We set $\delta_2 = 2h(2 + (16h + 4)\xi_{2h} + 2\xi_{2h})\eta(H)$, and $\delta_1 = \frac{\alpha}{k}$. Since $\alpha = O(k)$, we have that δ_1 is a constant; indeed, $\alpha \leq \Gamma = (2 + (16h + 4)\xi_{2h} + 2\xi_{2h})\eta(H)k$.

Since M^* is connected, we have that for every tree M_i there is a unique node in M_i that is incident with edges in $\mathcal{F}$. We denote this special node by r_i. We root the tree M_i at r_i. Let w be a child of r_i in M and let M_w and M_{r_i} be the subtrees of M obtained after deleting the edge $r_i w$. Since at least one edge incident with r_i is heavy, we have that $\mu(M_{r_i}, D) \geq h + 1$. However, the edge $r_i w$ is not heavy, and thus it must be the case that $\mu(M_w, D) \leq h$. Let $W = \Psi(M_w)$. Then, by Lemma 4.9, we have that $(D \cap W) \cup \kappa(r_i w)$ is a dominating set of size at most $2h$ for $G[W]$. Furthermore, the only vertices in W that have neighbors in $V(G) \setminus W$ belong to $\kappa(r_i w)$, and thus its size is also upper bounded by h. This implies that if $|W| > \xi_{2h}$, then it is a $2h$-DS-protrusion of size at least ξ_{2h}. Thus, from now onwards, we assume that this is not the case. This implies that, for every subtree rooted at r_i and every child w of r_i, we have that $|W = \Psi(M_w)| \leq \xi_{2h}$. Next, we look at $\tau(r_i)$ and based on its type. Recall from Theorem 2.4 that they are of the following types.

Case 1: $\tau(r_i)$ has at most h vertices of degree larger than h. In this case, we show that there exists an h^* depending only on h such that either $\tau(M_i)$ has at most $h^* = \xi_{h+\xi_{2h}} + h$ vertices of degree larger than h^* or G contains an h'-protrusion of size more than $\xi_{h'}$. Here $h' = \xi_{2h} + h$. Suppose some vertex v in $\tau(r_i)$ has degree at most h in $\tau(r_i)$ but has degree at least h^* in $\tau(M_i)$. Let N be the closed neighbourhood of v in $\tau(r_i)$ and N' be the neighborhood of v in $\tau(M_i)$. Each vertex in $N' \setminus N$ must lie in a connected component C of $\tau(M_i) \setminus N$ on at most ξ_{2h} vertices. Towards this end, observe that no vertex in C sees any vertex outside N even in the graph G. Thus, if $|C| > \xi_{2h}$, then we will get $2h$-DS-protrusion. Let X be N plus the union of all such components. By assumption, $|N' \setminus N| \geq \xi_{h+\xi_{2h}}$ and hence $|X| \geq \xi_{h+\xi_{2h}}$. Finally, the only vertices in X that have neighbors outside of X in G are in N, and $|N| \leq h$. The treewidth of $G[X]$ is at most $\xi_{2h} + h$, since removing N from X leaves components of size ξ_{2h}. Thus X is an h'-protrusion of size more than $\xi_{h'}$. If no such X exists, then it follows that every vertex of degree at most h in $\tau(r_i)$ has degree at most h^* in $\tau(M_i)$. The vertices of $\tau(M_i)$ that are not in $\tau(r_i)$ have degree at most $\xi_{2h} + h < h^*$. Thus $\tau(M_i)$ has at most $h < h^*$ vertices of degree at least h^*.

Case 2: $\tau(r_i)$ is h-nearly embedded in a surface Σ in which H cannot be embedded. In this case, we have that $\tau(r_i)$ excludes some graph H' depending only on h as a minor. The graph $\tau(M_i)$ can be obtained from $\tau(r_i)$ by joining constant size graphs (of size at most ξ_{2h}) to vertex sets that form cliques in $\tau(r_i)$. Thus there exists a graph H^* depending only on h such that $\tau(M_i)$ excludes H^* as a minor. This completes the proof of this lemma. □

5 Kernelization Algorithm for DS

In this section, we use the slice decomposition obtained in the last section to obtain linear kernels for DS, and in the next section we outline an algorithm for CDS.

Given an instance (G, k) of DS, we first apply Lemma 2.5 and find a dominating set D of G. If $|D| > \eta(H)k$, then we return that (G, k) is a NO instance of DS. Else, we apply Lemma 4.11 and

- either find a $(\delta_1 k, \delta_2 k)$-slice decomposition,
- a $2h$-DS-protrusion X of G of size more than ξ_{2h}, or
- a h'-protrusion of size more than $\xi_{h'}$ where h' depends only on h.

In the second case, we apply Lemma 4.3. Given X, by making use of Lemma 4.3, we obtain a boundaried graph X' such that $|X'| \leq \xi_{2h}$ and $X \equiv_{\mathrm{DS}} X'$. We also compute the translation constant c between X and X'. Now we replace the graph X with X' and obtain a new equivalent instance $(G', k+c)$. See Definition 3.4 for the notion of replacement. (Recall that c is a non-positive integer.) In the third case, we apply the protrusion replacement lemma of Reference [12, Lemma 7] to obtain a new equivalent instance (G', k') for $k' \leq k$ with $|V(G')| < |V(G)|$. We repeat this process until Lemma 4.11 returns a slice decomposition. For simplicity, we denote by (G, k) itself the graph on which Lemma 4.11 returns the slice decomposition. Since the number of times this process can be repeated is upper bounded by $n = |V(G)|$, we can obtain a $(\delta_1 k, \delta_2 k)$-slice decomposition for (G, k) in polynomial time.

Let $\mathcal{P}$ be the pairwise vertex disjoint subtrees $\{M_1, \ldots, M_\alpha\}$ of M coming from the slice decomposition of G. Recall that $R_i^+ = \Psi(M_i)$. Let $Q_i = \bigcup_{e \in \mathcal{E}(M_i)} \kappa(e)$, $B_i = (D \cap R_i^+) \cup Q_i$, and $b_i = |B_i|$. In this section, we will treat $G_i := G[R_i^+]$ as a graph with boundary B_i. Observe that by Lemma 4.9, it follows that B_i is a dominating set for G_i.

We have two kinds of graphs G_i. In one case, we have that G_i is H^*-minor-free for a graph H^* whose size depends only on h. In the other case, we have that the graph G_i has at most h' vertices of degree at least h'. To obtain our kernel, we will show the following two lemmas:

Lemma 5.1. *There exists a constant δ such that if G is a graph with boundary S such that S is a dominating set for G and G has at most h' vertices of degree at least h', then in polynomial time, we can obtain a graph G' with boundary S such that*

$$G' \equiv_{\mathrm{DS}} G \text{ and } |V(G')| \leq \delta |S|.$$

Furthermore, we can also compute the translation constant c of G and G' in polynomial time.

The second lemma is for H-minor-free graphs.

Lemma 5.2. *There exists a constant δ such that given an H-minor-free graph G with boundary S such that S is a dominating set for G we can obtain, in polynomial time, a graph G' with boundary S such that*

$$G' \equiv_{\mathrm{DS}} G \text{ and } |V(G')| \leq \delta |S|.$$

Furthermore, we can also compute the translation constant c of G and G' in polynomial time.

Once we have proved Lemmas 5.1 and 5.2, we construct the linear sized kernel for DS as follows. Given the graph G we obtain the slice decomposition and check if any of G_i has size more than δb_i. (Recall that $B_i = (D \cap R_i^+) \cup Q_i$ and $b_i = |B_i|$.) If yes, then we either apply Lemma 5.1 or Lemma 5.2 based on the type of G_i and obtain a graph G_i' such that $G_i' \equiv_{\mathrm{DS}} G_i$ and $|V(G_i')| \leq \delta b_i$. We think $G = G_i \oplus G^\star$, where $G^\star = G \setminus (R_i^+ \setminus B_i)$ as a b_i-boundaried graph with boundary B_i. Then we obtain a smaller equivalent graph $G' = G^\star \oplus G_i'$ and $k' = k + c$. After this, we can repeat the whole process once again. This implies that when we cannot apply Lemmas 5.2 or 5.1 on (G, k), we have that each of $|V(G_i)| \leq \delta b_i$. Furthermore, notice that by the definition of the slice decomposition, we have that $\cup_{i=1}^{\alpha} R_i^+ = V(G)$. This implies that in this case we have the following:

$$\begin{aligned}\sum_{i=1}^{\alpha} |R_i^+| \leq \delta \sum_{i=1}^{\alpha} b_i &= \delta \left(\sum_{i=1}^{\alpha} (|Q_i| + |(D \cap R_i^+) \setminus Q_i|) \right) \\ &= \delta \left(\sum_{i=1}^{\alpha} |Q_i| + \sum_{i=1}^{\alpha} |(D \cap R_i^+) \setminus Q_i| \right) \leq \delta\delta_2 k + \delta\eta(H)k = O(k).\end{aligned}$$

The last inequality follows from the fact that $\sum_{i=1}^{\alpha} |Q_i|$ is upper bounded by the second component of the slice decomposition and $\sum_{i=1}^{\alpha} |(D \cap R_i^+) \setminus Q_i|)$ is upper bounded by the size of the approximate dominating set D. This brings us to the following theorem:

Theorem 5.3. *DS admits a linear kernel on graphs excluding a fixed graph H as a topological minor.*

It only remains to prove Lemmas 5.1 and 5.2 to complete the proof of Theorem 5.3.

5.1 Irrelevant Vertex Rule and proofs for Lemmas 5.1 and 5.2

For the proofs of Lemmas 5.1 and 5.2, we will introduce a reduction rule that removes irrelevant vertices. If the graph G is $K_{h'}$-minor-free, then the irrelevant vertex rule will be used in a recursive fashion. In each recursive step, it is used to reduce the treewidth of torsos and hence also the entire graph. Then the graph is split into two pieces and the procedure is applied recursively to the two pieces. In the leaf of the recursion tree, when the graph becomes smaller, but is still big enough, then we apply Lemma 4.3 on it and obtain an equivalent instance.

Let G be a graph given with its tree-decomposition (M, Ψ) as described in Theorem 2.4 and $\tau(t)$ be one of its torsos. Let S be a dominating set of G, and $Z_t = A$, $|A| \leq h$, be the set of apices of $\tau(t)$. The reduction rule essentially "preserves" all dominating sets of size at most $|S|$ in G, without introducing any new ones. To describe the reduction rule, we need several definitions. The first step in our reduction rule is to classify different subsets A' of A into feasible and infeasible sets. The intuition behind the definition is that a subset A' of A is feasible if there exists a set D in G of size at most $|S| + 1$ such that D dominates all but S and $D \cap A = A'$. However, we cannot test in polynomial time whether such a set D exists. We will therefore say that a subset A' of A is *feasible* if the $\eta(H)$-approximation for DS (Lemma 2.5) outputs a set D of size at most $\eta(H)(|S| + 2)$ such that D dominates $V(G) \setminus (A \cup S)$ and $D \cap A = A'$. Observe that if such a set D of size at most $|S| + 1$ exists, then A' is surely feasible in the first sense, while if no such set D of size at most $\eta(H)(|S| + 2)$ exists, then A' is surely not feasible (again in the first sense). We will frequently use this in our arguments. Let us remark that there always exists a feasible set $A' \subseteq A$. In particular, $A' = S \cap A$ is feasible, since S dominates G. For feasible sets A', we will denote by $D(A')$ the set D output by the approximation algorithm.

For every subset $A' \subseteq A$, we select a vertex v of G such that $A' \subseteq N_G[v]$. If such a vertex exists, then we call it a *representative* of A'. Let us remark that some sets can have no representatives and some distinct subsets of A may have the same representative. We define R to be the set of representative vertices for subsets of A. The size of R is at most $2^{|A|}$. For $A' \subseteq A$, the set of *dominated vertices* (by A') is $W(A') = N(A') \setminus A$. We say that a vertex $v \in V(G) \setminus A$ is *fully dominated* by A' if $N[v] \setminus A \subseteq W(A')$. A vertex $w \in V(G) \setminus A$ is *irrelevant with respect to* A' if $w \notin R$, $w \notin S$, and w is fully dominated by A'.

Now we are ready to state the irrelevant vertex rule.

> **Irrelevant Vertex Rule:** If a vertex w is irrelevant with respect to every feasible $A' \subseteq A$, then delete w from G.

Lemma 5.4. *Let S be a dominating set in G, and G' be the graph obtained by applying the Irrelevant Vertex Rule on G, where w was the deleted vertex. Then $G' \equiv_{\mathrm{DS}} G$.*

Proof. We view G and G' as graphs with boundary S. Let the transposition constant be 0. To prove that $G' \equiv_{\mathrm{DS}} G$, we show that given a $|S|$-boundaried graph G_1 and a positive integer ℓ, we have that $(G \oplus G_1, \ell) \in \mathrm{DS} \Leftrightarrow (G' \oplus G_1, \ell) \in \mathrm{DS}$. Let $Z \subset V(G \oplus G_1)$ be a dominating set for $G \oplus G_1$ of size at most ℓ. Let $Z_1 = V(G) \cap Z$. If $|Z_1| > |S|$, then $(Z \setminus Z_1) \cup S$ is a smaller dominating set for $G \oplus G_1$. Therefore, we assume that $|Z_1| \leq |S|$. Let $A' = Z \cap A$, and observe that A' is feasible, because Z_1 dominates all but S. If $w \notin Z$, then $Z' = Z$ is a dominating set of size at most ℓ for $G' \oplus G_1$. So assume $w \in Z$. Observe that $w \in Z_1$ and $w \notin S$ and therefore all the neighbors of w lie in G. Since w is irrelevant with respect to all feasible subsets of A and A' is feasible, we have that

w is irrelevant with respect to A'. Hence $N_{G\oplus G_1}(w) \setminus N_{G\oplus G_1}(Z \setminus w) \subseteq A$. There is a representative $w' \in R, w' \neq w$ (since $w \notin R$), such that $(N_{G\oplus G_1}(w) = N_G(w)) \cap A \subseteq N_G(w') \cap A$. Hence $Z' = (Z \cup \{w'\}) \setminus \{w\}$ is a dominating set of $G' \oplus G_1$ of size at most ℓ.

Now, let $Z' \subseteq V(G' \oplus G_1)$ be a dominating set of size at most ℓ for $G' \oplus G_1$. Let $Z'_1 = V(G') \cap Z'$. As in the forward direction, we can assume that $|Z'_1| \leq |S|$. We show that Z' also dominates w in $G \oplus G_1$. Specifically $Z'_1 \cup \{w\}$ is a set dominating all but S in G of size at most $|S| + 1$ so $Z'_1 \cap A$ is feasible. Since $\{w\}$ is irrelevant with respect to $Z'_1 \cap A$, we have $w \in N_G(Z'_1 \cap A)$, and thus Z' is a dominating set for $G \oplus G_1$ of size at most ℓ. This concludes the proof. □

For a graph G and its dominating set S, we apply the Irrelevant Vertex Rule exhaustively on all torsos of G, obtaining an induced subgraph G' of G. By Lemma 5.4 and transitivity of $\equiv_{\mathrm{DS}}$, we have that $G' \equiv_{\mathrm{DS}} G$. We now prove that a graph G that cannot be reduced by the irrelevant vertex rule has a property that each of its torso has a small 2-dominating set.

Lemma 5.5. *Let G be a graph that is irreducible by the Irrelevant Vertex Rule and S be a dominating set of G. For every torso $\tau(t)$ of the tree-decomposition (M, Ψ) of G, we have that $\tau(t) \setminus Z_t$ has a 2-dominating set of size $O(|S|)$. Furthermore, if G is a H-minor-free graph, then $\mathbf{tw}(G) = O(\sqrt{|S|})$.*

Proof. Let $\tau(t)^* = \tau(t) \setminus A$, where A are the apices of $\tau(t)$. We will obtain a 2-dominating set of size $O(|S|)$ in $\tau(t)^*$. Towards this end, consider the following set:

$$Q = \bigcup_{A' \subseteq A,\, A' \text{is feasible}} D(A') \cup R \cup (S \setminus A).$$

The number of representative vertices R and the number of feasible subsets A' is at most $2^{|A|} \leq 2^h$, where h is a constant depending only on H. The size of $D(A')$ is at most $\eta(H)(|S| + 2)$ for every A'. Thus $|Q| \leq 2^h(\eta(H)(|S| + 2)) + 2^h + |S| = O(|S|)$. We prove that Q is a 2-dominating set of $V(G) \setminus A$. Let $w \in V(G) \setminus A$. If $w \in R$ or $w \in S$, then Q dominates w. So suppose $w \notin R \cup S$. Then, since w is not irrelevant, we have that there is a feasible subset A' of A such that w is relevant with respect to A'. Hence w is not fully dominated by A' and so w has a neighbour $w' \in V(G) \setminus N[A']$. But w' is dominated by $D(A') \subseteq Q$, and thus w is 2-dominated by Q in $G \setminus A$. Hence, $G \setminus A$ has a 2-dominating set of size $O(|S|)$.

The graph $\tau(t)^*$ can be obtained from $G \setminus A$ by contracting all edges in $E(G \setminus A) \setminus E(\tau(t)^*)$ and adding all edges in $E(\tau(t)^*) \setminus E(G \setminus A)$. Since contracting and adding edges does not increase the size of a minimum 2-dominating set of a graph, $\tau(t)^*$ has a 2-dominating set of size $O(|S|)$. This completes the proof for the first part.

Now assume that G is a H-minor-free graph. It is well known that the treewidth of a H-minor-free graph is at most the maximum treewidth of its torsos, see, for example, Reference [21]. Thus to show that $\mathbf{tw}(G) = O(\sqrt{|S|})$ it is sufficient to show that its torsos have small treewidth. To conclude, $\tau(t)^*$ excludes an apex graph as a minor (see, for example, Reference [46, Theorem 13]) and it has a 2-dominating set of size $O(|S|)$. By the bidimensionality of 2-dominating set, we have that $\mathbf{tw}(\tau(t)^*) = O(\sqrt{|S|})$ [21, 37]. Now we add all the apices of A to all the bags of the tree-decomposition of $\tau(t)^*$ to obtain a tree-decomposition for $\tau(t)$ of width $O(\sqrt{|S|}) + h = O(\sqrt{|S|})$. □

Let us also remark that Irrelevant Vertex Rule is based on the performance of a polynomial time approximation algorithm. Thus by Lemmas 2.5, 5.4, and 5.5, and the fact that the treewidth of a graph is at most the maximum treewidth of its torsos, see for example, Reference [21], we obtain the following lemma:

Lemma 5.6. *There is a polynomial time algorithm that for a given graph G and a dominating set S of G, outputs graph G' such that $G' \equiv_{\mathrm{DS}} G$ and for every torso $\tau(t)$ of the tree-decomposition (M, Ψ)*

of G, we have that $\tau(t) \setminus Z_t$ has a 2-dominating set of size $O(|S|)$. Furthermore, if G is a H-minor-free graph, then $\mathbf{tw}(G) = O(\sqrt{|S|})$.

Before we proceed further, we show the power of Lemma 5.6 by deriving a simple subexponential time algorithm for DS on H-minor-free graph. This is one of the cornerstone results in Reference [21] and is based on a non-trivial two-layer dynamic programming over clique-sum decomposition tree of a H-minor-free graphs. Lemma 5.6 can be used to obtain much simpler algorithm. Given a graph G and a positive integer k, we first apply a factor 2-approximation algorithm given in References [22, 39] for DS on G and obtain a set S. If the size of S is more than $2k$, then we return that G does not have a dominating set of size at most k. Otherwise, we apply Lemma 5.6 and obtain an equivalent graph G' such that $\mathbf{tw}(G') = O(\sqrt{k})$. Now applying a constant factor approximation algorithm developed in Reference [21] for computing the treewidth on G', we get a tree-decomposition of width $O(\sqrt{k})$. It is well known that checking whether a graph with treewidth t has a dominating set of size at most k can be done in time $O(3^t n^{O(1)})$ [59]. This, together with the preceding bound on the treewidth, gives us an alternative proof of the following theorem:

Theorem 5.7 ([22]). *Given an n-vertex graph G excluding a fixed graph H as a minor, one can check whether G has a dominating set of size at most k in time $2^{O(\sqrt{k})}n^{O(1)}$.*

Having Lemma 5.6 proving Lemma 5.1 becomes simple.

Proof of Lemma 5.1. We apply Lemma 5.6 to G with a decomposition that has a single bag containing the entire graph and the apices A of the bag being the vertices of degree at least h'. By Lemma 5.6, $G \setminus A$ has a 2-dominating set of size $\delta_3|S|$. Since all vertices of $G \setminus A$ have degree at most h', it follows that $|V(G)| \le h' + \delta_3|S| + \delta_3 h|S| + \delta_3 h^2|S| \le \delta|S|$. □

We need the following well-known lemma, see, for example, Reference [9], on separators in graphs of bounded treewidth for the proof of Lemma 5.2.

Lemma 5.8. *Let G be a graph given with a tree-decomposition of width at most t and $w : V(G) \to \{0, 1\}$ be a weight function. Then, in polynomial time, we can find a bag X of the given tree-decomposition such that for every connected component $G[C]$ of $G \setminus X$, $w(C) \le w(V(G))/2$. Furthermore, the connected components $C_1, \ldots, C_\ell$ of $G \setminus X$ can be grouped into two sets V_1 and V_2 such that $\frac{w(V(G))-w(X)}{3} \le w(V_i) \le \frac{2(w(V(G))-w(X))}{3}$ for $i \in \{1, 2\}$.*

Proof of Lemma 5.2. By (G, S), we denote the graph with boundary S. By Lemma 5.6, we may assume that $\mathbf{tw}(G) = O(\sqrt{|S|})$. We prove the lemma using induction on $|S|$. If $|S| = O(1)$, then we are done, as, in this case, we know that G is a $|S|$-DS protrusion. Thus, if $|V(G)| > \xi_{|S|}$, then we can apply Lemma 4.3 and in polynomial time obtain a graph G^* such that $G^* \equiv_{\mathrm{DS}} G$ and $|V(G^*)| \le \xi_{|S|}$. At the same time, we can compute the translation constant depending on G and G^* and return it. Thus, we return G^* and the translation constant c.

Otherwise, using a constant factor approximation of treewidth on H-minor-free graphs [34], we compute a tree-decomposition of G of width $d\sqrt{|S|}$ for some constant d. Now, by applying Lemma 5.8 on this decomposition, we find a partitioning of $V(G)$ into V_1, V_2, and X such that there are no edges from V_1 to V_2, $|X| \le d\sqrt{|S|} + 1$, and $|V_i \cap S| \le 2|S|/3$ for $i \in \{1, 2\}$. Let $S' = S \cup X$. Observe that S' is also a dominating set.

Let $S_1 = S' \cap (V_1 \cup X)$ and $S_2 = S' \cap (V_2 \cup X)$. Let $G_1 = G[V_1 \cup X]$ and $G_2 = G[V_2 \cup X]$. We now apply the algorithm recursively on (G_1, S_1) and (G_2, S_2) and obtain graphs G_1', G_2' such that for $i \in \{1, 2\}$, $G_i \equiv_{\mathrm{DS}} G_i$. Let c_1 and c_2 be the translation constants returned by the algorithm. Since $X \subseteq S'$, we have that S_i is a dominating set of G_i, and hence we actually can run the algorithm recursively on the two subcases. The algorithm returns G'_1 and G'_2 and translation constants c_1 and c_2. Let $G' = G'_1 \oplus_\delta G'_2$ and $S' = S_1 \cup S_2$. We will show that $G' \equiv_{\mathrm{DS}} G$. Let G_3 be a graph with boundary S' and k be a positive integer. Then

$$\begin{aligned}
&((G_1 \oplus_\delta G_2) \oplus G_3, k) \in \mathrm{DS} \\
\iff &((G_1 \oplus_\delta G_3) \oplus G_2, k) \in \mathrm{DS} \\
\iff &((G_1 \oplus_\delta G_3) \oplus G'_2, k + c_2) \in \mathrm{DS} \\
\iff &((G'_2 \oplus_\delta G_3) \oplus G_1, k + c_2) \in \mathrm{DS} \\
\iff &((G'_2 \oplus_\delta G_3) \oplus G'_1, k + c_2 + c_1) \in \mathrm{DS} \\
\iff &((G'_2 \oplus_\delta G'_1) \oplus G_3, k + c_2 + c_1) \in \mathrm{DS}.
\end{aligned}$$

This proves that $G' \equiv_{\mathrm{DS}} G$. Now we will show that $|V(G')| \leq O(|S|)$.

Let $\mu(|S|)$ be the largest possible size of the set $|V(G')|$ output by the algorithm when run on a graph G with a dominating set S. We upper bound $|V(G')|$ by the following recursive formula:

$$|V(G')| \leq \max_{1/3 \leq \alpha \leq 2/3} \left\{ \mu\left(\alpha|S| + d\sqrt{|S|}\right) + \mu\left((1-\alpha)|S|\right) + d\sqrt{|S|} \right\}.$$

Using simple induction, one can show that the preceding solves to $O(|S|)$. See for an example Reference [39, Lemma 2]. Hence, we conclude that $|V(G')| = O(|S|) = O(k)$. This completes the proof of the lemma. □

The algorithm of Demaine et al. [22] computing a dominating set of size k in an n-vertex H-minor-free graph uses exponential (in k) space $2^{O(\sqrt{k})} n^{O(1)}$. Theorem 5.3 implies almost directly the following refinement of Theorem 5.7.

Theorem 5.9. *Given an n-vertex graph G excluding a fixed graph H as a minor, one can check whether G has a dominating set of size at most k in time $2^{O(\sqrt{k})} + n^{O(1)}$ and space $(nk)^{O(1)}$.*

Proof. Our algorithm first applies Theorem 5.3 to obtain a graph with $O(k)$ vertices. Now we are assuming that the number of vertices in G is $n = O(k)$. We solve a slightly more general version of domination, where we are given a subset S and the requirement is to find a set D of size at most k such that for every $v \in V(G) \setminus S$, $N[v] \cap D \neq \emptyset$. When $S = \emptyset$, the set D is a dominating set of size k. By the separator theorem of Alon et al. [6] for H-minor-free graphs, one can find in polynomial time a partition of $V(G)$ into V_1, V_2, and X such that $|X| \leq O(\sqrt{n})$, and there are no edges from V_1 to V_2 and $|V_i| \leq 2n/3$ for $i \in \{1, 2\}$. The algorithm finds such a partition and guesses how D interacts with X.

In particular, first the algorithm correctly guesses $D' = D \cap X$ (by looping over all subsets of X). For each guess, it puts $N(D')$ into S and removes D' and $S \cap X$ from G (these vertices are already dominated and will not be used in the future to dominate even more vertices). For every remaining vertex v in X, the algorithm guesses whether it will be dominated by a vertex in V_1, in which case the algorithm deletes all edges from v to vertices in V_2, or by a vertex in V_2, in which case the algorithm deletes all edges from v to vertices in V_1. Let V'_i be V_i plus all the vertices in $X \setminus S$ that we guessed were dominated from V_i. At this point, V'_1 and V'_2 are distinct components of the instance and can be solved independently. The running time is governed by the following recurrence:

$$T(n) = n^{O(1)} \cdot 2^{O(\sqrt{n})} \cdot 2 \cdot T(2n/3) = 2^{O(\sqrt{n})}.$$

The space used is clearly polynomial. This concludes the proof. □

6 KERNELIZATION ALGORITHM FOR CDS

The kernelization algorithm for CDS is similar to DS—we also use slice decomposition to obtain a linear kernel. However, the irrelevant vertex rule is a bit different. The kernelization algorithm for CDS follows from the results analogous to Lemmas 5.2 and 5.1 for DS. For completeness, we spell out all the steps.

In particular, given an instance (G, k) of CDS, we first apply Lemma 2.5 and find a dominating set D of G. If $|D| > \eta(H)k$, then we return that (G, k) is a NO instance of CDS. Else, we apply Lemma 4.11 and

- either find $(\delta_1 k, \delta_2 k)$-slice decomposition,
- a $2h$-CDS-protrusion of size more than ξ_{2h}, or
- a h'-protrusion of size more than $\xi_{h'}$ where h' depends only on h.

In the second case, we apply Lemma 4.3. For a given X, we apply Lemma 4.3 and construct a boundaried graph X' such that $|X'| \le \xi_{2h}$ and $X \equiv_{\text{CDS}} X'$. We also compute the translation constant c between X and X'. Now we replace the graph X with X' and obtain a new equivalent instance $(G', k + c)$, here we remind that c is a non-positive integer. In the third case we apply the protrusion replacement lemma of Reference [12, Lemma 7] to obtain a new equivalent instance (G', k') for $k' \le k$ with $|V(G')| < |V(G)|$. We repeat this process until Lemma 4.11 returns a slice decomposition. For simplicity, we denote by (G, k) itself the graph on which Lemma 4.11 returns the slice decomposition. The number of times this process can be repeated does not exceed $n = |V(G)|$, and a $(\delta_1 k, \delta_2 k)$-slice decomposition for (G, k) is constructed in polynomial time.

The pairwise disjoint connected subtrees $\{M_1, \ldots, M_\alpha\}$ of M coming from the slice decomposition of G is denoted by $\mathcal{P}$, and we put $R_i^+ = \Psi(M_i)$. We define $Q_i = \bigcup_{e \in \mathcal{E}(M_i)} \kappa(e)$, $B_i = (D \cap R_i^+) \cup Q_i$, and $b_i = |B_i|$. As in the previous section, we treat $G_i := G[R_i^+]$ as a graph with boundary B_i. Then, by Lemma 4.9, B_i is a dominating set for G_i.

For two kinds of graphs G_i, we use different reductions. In the first case, we have that the graph G_i has at most h' vertices of degree at least h'.

Lemma 6.1. *There exists a constant δ such that if G is a graph with boundary S such that S is a dominating set for G and G has at most h' vertices of degree at least h', and then, in polynomial time, we can obtain a graph G' with boundary S such that*

$$G' \equiv_{\text{CDS}} G \text{ and } |V(G')| \le \delta|S|.$$

Furthermore, we can also compute the translation constant c of G and G' in polynomial time.

In the other case, we have that G_i is H^*-minor-free for a graph H^* whose size only depends on h.

Lemma 6.2. *There exists a constant δ such that given an H-minor-free graph G with boundary S such that S is a dominating set for G, in polynomial time, we can obtain a graph G' with boundary S such that*

$$G' \equiv_{\text{CDS}} G \text{ and } |V(G')| \le \delta|S|.$$

Furthermore, we can also compute the translation constant c of G and G' in polynomial time.

To obtain the linear sized kernel for CDS, the proof of Lemmas 6.1 and 6.2 suffices. Indeed, for graph G, we obtain the slice decomposition and check if any G_i has size more than δb_i. If yes, then we either apply Lemma 6.1 or Lemma 6.2 based on the type of G_i and obtain a graph G_i' such that $G_i' \equiv_{\text{CDS}} G_i$ and $|V(G_i')| \le \delta b_i$. We view $G = G_i \oplus G^\star$, where $G^\star = G \setminus (R_i^+ \setminus B_i)$ as a b_i-boundaried graph with boundary B_i. Then we obtain a smaller equivalent graph $G' = G^\star \oplus G_i'$ and $k' = k + c$. After this, we can repeat the whole process once again. This implies that when we cannot apply Lemmas 6.2 or 6.1 on (G, k), we have that each of $|V(G_i)| \leq \delta b_i$. Furthermore, notice that $\cup_{i=1}^{\alpha} R_i^+ = V(G)$. This implies that

$$\sum_{i=1}^{\alpha} |R_i^+| \leq \delta \sum_{i=1}^{\alpha} b_i = \delta \left(\sum_{i=1}^{\alpha} (|Q_i| + |(D \cap R_i^+) \setminus Q_i|) \right)$$
$$= \delta \left(\sum_{i=1}^{\alpha} |Q_i| + \sum_{i=1}^{\alpha} |(D \cap R_i^+) \setminus Q_i| \right) \leq \delta\delta_2 k + \delta\eta(H)k = O(k).$$

Thus (subject to the proof of two lemmas) we have the following theorem:

Theorem 6.3. *CDS admits a linear kernel on graphs excluding a fixed graph H as a topological minor.*

6.1 Irrelevant Vertex Rule and Proofs for Lemmas 6.1 and 6.2

As with DS, we will reduce the treewidth of a torso not only in the beginning of the procedure but also when we apply it recursively. Let G be an H-minor-free graph, S be a dominating set of G (not necessarily connected), $\tau(t)$ be one of its torsos, and A, $|A| \leq h$, be the set of apices of $\tau(t)$, where h is some constant depending only on H. We will define a reduction rule that essentially "preserves" all dominating sets of size at most $3|S| + 3$ with "good-enough" connectivity properties, without introducing new such sets. Just as for DS, we will say that a subset A' of A is feasible if the factor $\eta(H)$-approximation for DS (Lemma 2.5) concludes that there exists a set D of size at most $\eta(H)(3|S| + 3)$ that dominates $V(G) \setminus (A \cup S)$ and $D \cap A = A'$. If such a set exists and A' is feasible, then we denote this set by $D(A')$.

Recall that for DS we had the notion of a representative element for every subset $A' \subseteq A$. The representative vertex was crucially used in establishing Lemma 5.4, where we used it to simulate all the domination properties of the deleted vertex "w." We need a similar notion of representatives for CDS; however, here the representatives will be vertex subsets rather than single vertices. With vertex subsets, we will be able to simulate not only domination properties but also the connectivity properties of an irrelevant vertex. More precisely, for every subset $A' \subseteq A$, we compute a minimum size vertex set $T \subseteq V(G) \setminus A$ such that $G[T]$ is connected and $A' \subseteq N[T]$. If the size of such a minimum set is at most $4h$, then we say that $T = T(A')$ is a *representative* of A', and add all the vertices in T to the set R. Note that $|R| \leq 4h \cdot 2^h$. For each A' we can test whether a representative exists in time $2^{|A'|} n^{O(1)} = 2^h n^{O(1)}$ by making a modification of the algorithm for the Steiner tree problem from Reference [8]. Alternatively, we can test it in time $n^{4h+O(1)}$ by brute force. Let S_{4h} denote the set of vertices in $N^{4h}_{G\setminus A}[S] = N^{4h}_{G\setminus A}[S \setminus A]$. Here $N^{4h}_{G\setminus A}[w]$ is the set of vertices at distance at most $4h$ from w in the graph $G \setminus A$ (not in G). The set of vertices *covered* by A' is $W(A') = N[A'] \setminus (A \cup S \cup S_{4h})$. Note that a vertex in $N^{4h}_{G\setminus A}[S]$ is never covered by a set A'. Let CutVert denote the set of vertices w in G such that $G - \{w\}$ has more connected components than G. Observe that if G will be connected then CutVert is essentially the set of cut vertices. However, for a disconnected graph, it is the union of cut vertices for each connected component.

The definition of an irrelevant vertex with respect to A is different than for DS. A vertex

$$w \notin (S \cup S_{4h} \cup R \cup \mathrm{CutVert})$$

is called *irrelevant with respect to* A' if $N^{4h}_{G\setminus A}[w] \subseteq W(A')$. The irrelevant vertex rule for CDS is exactly the same as in Section 5 for DS but the correctness proof and analysis is more complicated. Recall that a subset A' of A is feasible if the factor $\eta(H)$-approximation for DS (Lemma 2.5) concludes that there exists a set D of size at most $\eta(H)(3|S| + 3)$ which dominates all but S, such that $S \cap A = A'$.

Irrelevant Vertex Rule: If a vertex w is irrelevant with respect to every feasible $A' \subseteq A$, then delete w from G.

Lemma 6.4. *Let S be a dominating set in G, and G' be the graph obtained by applying the Irrelevant Vertex Rule on G, where w was the deleted vertex. Then $G' \equiv_{\mathrm{CDS}} G$.*

Proof. We view G and G' as graphs with boundary S. Let the transposition constant be 0. To show that $G' \equiv_{\mathrm{CDS}} G$, we show that, given any boundaried graph G_1 and a positive integer ℓ, we have that $(G \oplus G_1, \ell) \in \mathrm{CDS} \Leftrightarrow (G' \oplus G_1, \ell) \in \mathrm{CDS}$. Let $Z \subset V(G \oplus G_1)$ be a connected dominating set for $G \oplus G_1$ of size at most ℓ. Observe that since S is a dominating set of G, we have that there exists a connected dominating set $S \subseteq S^*$ such that $|S^*| \leq 3|S|$ (Proposition 2.6). Let $Z_1 = V(G) \cap Z$. If $|Z_1| > 3|S|$, then $(Z \setminus Z_1) \cup S^*$ is a smaller connected dominating set for $G \oplus G_1$. Thus, we assume that $|Z_1| \leq 3|S|$. Let $A' = Z_1 \cap A$, and observe that A' is feasible, since Z_1 dominates all but S and has size at most $3|S|$. If $w \notin Z$, then $Z' = Z$ is a connected dominating set of size ℓ for $G' \oplus G_1$. So assume $w \in Z$. Since w is irrelevant with respect to A', we have that $N^{4h}_{G \setminus A}[w] \subseteq W(A')$.

Let Q be the connected component of $G \oplus G_1$ that contains w. Since w is not a cut vertex of G, we have the following easy observation.

Observation 1. *$Q \setminus \{w\}$ is connected.*

Let $Z_Q = Z \cap Q$ be the connected dominating set of Q, $|Z_Q| = p$. We will show that $Q \setminus \{w\}$ has a connected dominating set of size at most p and that will show that $(G' \oplus G_1, \ell) \in \mathrm{CDS}$. Observe that since $w \in W(A')$ and the only vertices that are common between G and G_1 belong to S, we have that $N^{4h}_{(G \oplus G_1) \setminus A}[w] = N^{4h}_{G \setminus A}[w] \subseteq V(G) \setminus S_{3h}$.

Let X be the vertex set of the connected component of $G \oplus G_1[Z_Q \cap N^{4h}_{G \setminus A}[w]]$ that contains w. If $|X| < 4h$, then there is a subset $X' = T(N(X) \cap A)$ such that $X' \subset R$, $|X'| \leq |X|$, $G[X']$ is connected and $N_G(X') \cap A \supseteq N_G(X) \cap A$. Furthermore, since $|X| < 4h$, we have that every connected component of $G \oplus G_1[Z_Q \setminus X]$ contains a vertex of A'. This implies that $Z'_Q = (Z_Q \setminus X) \cup X'$ is connected. Since $X \subseteq W(A')$ and $|X| < 4h$, we have that $N_{G \oplus G_1}(X) = N_G(X)$. This implies that $N_G(X) \subseteq N_G(X' \cup A') \subseteq N_{G \oplus G_1}(X' \cup A')$ and thus Z'_Q is a connected dominating set of size at most p of Q that avoids w and thus by Observation 1, it is also a connected dominating set of $Q \setminus \{w\}$. This implies that in this case $(G' \oplus G_1, \ell) \in \mathrm{CDS}$.

Now suppose that $|X| \geq 4h$. Let $A^* = N_G(X) \cap A$. The vertex set A^* is a dominating set of size at most h in the connected graph $G[A^* \cup X]$ and so $G[A^* \cup X]$ has a connected dominating set X^* that contains A^* of size at most $3h$. Let P be the connected component of $G[X^*] \setminus A$ that contains w. Notice that $|P| \leq 2h$, and so there is a connected set $P' \subseteq R$ such that $|P'| \leq |P|$ and $N(P) \cap A \subseteq N(P') \cap A$. Finally, let Y be the set of vertices in X that are at distance exactly $4h$ from w in $G \setminus A$. Note that $|X \setminus Y| \geq 4h - 1$ (as every path from w to a vertex in Y has length at least $4h - 1$) and that $N_G[Y] \cap A \subseteq A^*$. Set $X' = (X^* \setminus P) \cup P'$, and $Z'_Q = (Z_Q \setminus (X \setminus Y)) \cup X'$. We have that $|X'| \leq |X^*| \leq 3h$ while $|X \setminus Y| \geq 4h - 1 \geq 3h$. Hence $|Z'_Q| \leq |Z_Q|$. Note that $G[X']$ is connected. Furthermore, by our choice of $(X \setminus Y)$, we have that every connected component of $G \oplus G_1[Z_Q \setminus X]$ contains a vertex of Y and hence a vertex of A^*. However, $A^* \subseteq X'$ and $G[X']$ (or $G \oplus G_1[X']$) is connected, and thus $G \oplus G_1[Z'_Q]$ is connected. Observe that $N_{G \oplus G_1}(X \setminus Y) = N_G(X \setminus Y)$. This implies that $N_G(X \setminus Y) \subseteq N_G(X' \cup A^*) \subseteq N_{G \oplus G_1}(X' \cup A^*)$, and thus Z'_Q is a connected dominating set of size at most p of Q that avoids w and thus, by Observation 1, is also a connected dominating set of $Q \setminus \{w\}$. This implies that in this case $(G' \oplus G_1, \ell) \in \mathrm{CDS}$.

Now we prove the reverse direction. Let $Z' \subset V(G' \oplus G_1)$ be a connected dominating set for $G' \oplus G_1$ of size at most ℓ. By Observation 1, we know that $Q \setminus \{w\}$ and Q are connected and thus Z' is also a connected dominating set of size at most ℓ for $G \oplus G_1$. This concludes the proof. □

Next we prove an auxiliary lemma that upper bounds the number of cut vertices in terms of the dominating set of the graph.

Cuts and Blocks. A maximal connected subgraph without a cut vertex is called a **block**. Every block of a graph G is either a maximal 2-connected subgraph, a bridge, or an isolated vertex. By maximality, different blocks of G overlap in at most one vertex, which is then a cut vertex of G. Therefore, every edge of G lies in a unique block, and G is the union of its blocks.

Definition 6.5. Let A denote the set of cut vertices of G and B the set of its blocks. The bipartite graph on $A \cup B$ where $a \in A$ and $b \in B$ are adjacent when $a \in b$ is called the block graph of G.

Proposition 6.6 (reference [25]). *The block graph of a connected graph is a tree.*

Lemma 6.7. *Let G be a graph and S be a dominating set of G; then the number of cut vertices in G is upper bounded by $|S|$. That is,* $|\text{CutVert}| \leq |S|$.

Proof. Let $A = \text{CutVert}$ denote the set of cut vertices of G and B the set of its blocks. Consider the block graph $\mathcal{B}$ on $A \cup B$. By Proposition 6.6, we know that $\mathcal{B}$ is a tree. Now we root this tree at some vertex in B. Observe that there is unique association of cut vertices to its parent—which is a block. We also know that for every cut vertex v that either v is in S or a vertex in its parent block. However, the blocks are pairwise disjoint except for the vertices in A. Thus, this implies that there is an injective map from A to S and hence $|\text{CutVert}| \leq |S|$. ◻

Now we are ready to prove the treewidth bounding lemma of this section. Just as for DS, it is possible to prove that after removing all irrelevant vertices, the treewidth of each torso in the reduced graph is $O(\sqrt{|S|})$. The most important difference is that instead of 2-dominating set, we construct a $8h$-dominating set in the proof. We start with the following auxiliary lemma that will be useful for the proof.

Lemma 6.8. *There is a polynomial time algorithm that for a given graph G and a dominating set S of G, outputs graph G' such that $G' \equiv_{\text{CDS}} G$, and for every torso $\tau(t)$ of the tree-decomposition (M, Ψ) of G, we have that $\tau(t) \setminus Z_t$ has a $8h$-dominating set of size $O(|S|)$. Furthermore, if G is a H-minor-free graph, then* $\mathbf{tw}(G) = O(\sqrt{|S|})$.

Proof. Let $\tau(t)^* = \tau(t) \setminus A$, where A are the apices of $\tau(t)$. Also, let CutVert denote the set of cut vertices of G. We will obtain a $(4h+1)$-dominating set of size $O(|S|)$ in $\tau(t)^*$. Towards this end, consider the following set:

$$Q = \bigcup_{A' \subseteq A, A' \text{is feasible}} D(A') \cup R \cup (S \setminus A) \cup \text{CutVert}.$$

The size of the set of representative vertices, R, is at most $4h \cdot 2^{|A|} \leq 4h \cdot 2^h$. The number of feasible subsets A' is at most 2^h, where h is a constant depending only on H. The size of $D(A')$ is at most $\eta(H)(3|S|+3)$ for every A'. By Lemma 6.7, we have that $|\text{CutVert}| \leq |S|$. Thus $|Q| \leq 2^h(\eta(H)(3|S|+3)) + 4h \cdot 2^h + 2|S| = O(|S|)$. We prove that Q is a $(4h+1)$-dominating set of $V(G) \setminus A$. Let $w \in V(G) \setminus A$. If $w \in R$ or $w \in S$ or $w \in \text{CutVert}$, then Q dominates S. So suppose $w \notin R \cup S \cup \text{CutVert}$. Then, since w is not irrelevant, there is a feasible subset A' of A such that w is relevant with respect to A'. Hence there exists a vertex w' in $N^{4h}_{G \setminus A}[w]$ that is not in $W(A')$. If $w' \in S_{4h}$, S_{4h} denotes the set of vertices in $N^{4h}_{G \setminus A}[S] = N^{4h}_{G \setminus A}[S \setminus A]$, then w is $8h$-dominated by a vertex $w^* \in (S \setminus A) \subseteq Q$ in $G \setminus A$. Otherwise, w' is dominated by some w'' in $D(A')$, and hence w is $4h+1$-dominated by $w'' \in Q$ in $G \setminus A$. Hence $G \setminus A$ has a $8h$-dominating set of size $O(|S|)$.

The graph $\tau(t)^*$ can be obtained from $G \setminus A$ by contracting all edges in $E(G \setminus A) \setminus E(\tau(t)^*)$ and adding all edges in $E(\tau(t)^*) \setminus E(G \setminus A)$. Since contracting and adding edges cannot increase the size of a minimum $8h$-dominating set of a graph, $\tau(t)^*$ has a $8h$-dominating set of size $O(|S|)$. This completes the proof for the first part.

Now assume that G is a H-minor-free graph. It is well known that the treewidth of an H-minor-free graph is at most the maximum treewidth of its torsos, see for example, Reference [21]. Thus, to show that $\mathbf{tw}(G) = O(\sqrt{|S|})$, it is sufficient to show that its torsos have small treewidth. To conclude, $\tau(t)^*$ excludes an apex graph as a minor (see discussions after Theorem 2.4), and it has a $8h$-dominating set of size $O(|S|)$. By the bidimensionality of $8h$-dominating set, we have that $\mathbf{tw}(\tau(t)^*) = O(\sqrt{|S|})$ [21, 37]. Now we add all the apices of A to all the bags of the tree-decomposition of $\tau(t)^*$ to obtain a tree-decomposition for $\tau(t)'$. Thus $\mathbf{tw}(\tau(t)') \leq O(\sqrt{|S|}) + h = O(\sqrt{|S|})$.

Let us also remark that the Irrelevant Vertex Rule is based on the performance of a polynomial time approximation algorithm, and thus the whole procedure can be implemented in polynomial time. This concludes the proof. □

Having Lemma 6.8 proving Lemma 6.1 becomes simple.

Proof of Lemma 6.1. We apply Lemma 6.8 to G with a decomposition that has a single bag containing the entire graph and the apices A of the bag being the vertices of degree at least h'. By Lemma 6.8, $G \setminus A$ has a $8h$-dominating set of size $\delta_3|S|$. Since all vertices of $G \setminus A$ have degree at most h', it follows that $|V(G)| \leq h' + h'^{O(h')}\delta_3|S| \leq \delta|S|$. □

Proof for Lemma 6.2 is identical to the proof of Lemma 5.2, except that we need to use Lemma 6.8 in place of Lemma 5.6. Thus we omit it.

Recently, Bodlaender et al. [10] obtained an algorithm solving CDS on graphs of treewidth t in time $c^t n^{O(1)}$. Theorem 6.3 combined with this implies that CDS on H-minor-free graphs is solvable in time $2^{O(\sqrt{k})} + n^{O(1)}$. To our knowledge, this is the first subexponential parameterized algorithm for CDS on H-minor-free graphs.

Theorem 6.9. *Given an n-vertex graph G excluding a fixed graph H as a minor, one can check whether G has a connected dominating set of size at most k in time* $2^{O(\sqrt{k})} + n^{O(1)}$.

7 CONCLUSIONS

In this article, we give linear kernels for two widely studied parameterized problems, namely DS and CDS, for every graph class that excludes some graph as a topological minor. The emerging questions are the following:

(1) Can our kernelization results for DS and CDS be extended to more general sparse graph classes?
(2) Can our techniques be applied to more general families of parameterized problems?

Very recently, the first question was answered both positively and negatively by Drange et al. [29]. In particular, DS admits a vertex-linear kernel on graphs of bounded expansion and an almost vertex-linear kernel on nowhere-dense graphs. On the other hand, CDS admits no polynomial kernel on graphs of bounded expansion unless coNP$\subseteq$ NP/poly. It is important to point out that methods used by Drange et al. [29] is entirely different than ours. Their algorithm is completely combinatorial and do not rely on topological arguments. Our kernelization algorithm for CDS is still the best known. It would be interesting to see if the combinatorial methods developed in Drange et al. [29] could be used to design an explicit kernelization algorithm for CDS on graph classes excluding a fixed graph H as a topological minor.

ACKNOWLEDGMENTS

We thank Marek Cygan for sending us a copy of Reference [16]. We sincerely thank all the reviewers for their insightful comments and suggestions.

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Received October 2016; revised July 2017; accepted October 2017