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# On the parameterized complexity of $[1, j]$-domination problems

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For a graph $G$, a set $D \subseteq V(G)$ is called a $[1, j]$-dominating set if every vertex in $V(G) \setminus D$ has at least one and at most $j$ neighbors in $D$. A set $D \subseteq V(G)$ is called a $[1, j]$-total dominating set if every vertex in $V(G)$ has at least one and at most $j$ neighbors in $D$. In the $[1, j]$-(Total) Dominating Set problem we are given a graph $G$ and a positive integer $k$. The objective is to test whether there exists a $[1, j]$-(total) dominating set of size at most $k$. The $[1, j]$-Dominating Set problem is known to be NP-complete, even for restricted classes of graphs such as chordal and planar graphs, but polynomial-time solvable on split graphs. The $[1, 2]$-Total Dominating Set problem is known to be NP-complete, even for bipartite graphs. As both problems generalize the Dominating Set problem, both are W[1]-hard when parameterized by solution size. In this work, we study the aforementioned problems on various graph classes from the perspective of parameterized complexity and prove the following results:

- $[1, j]$-Dominating Set parameterized by solution size is W[1]-hard on $d$-degenerate graphs for $d = j + 1$.
- $[1, j]$-Dominating Set parameterized by solution size is FPT on nowhere dense graphs.
- The known algorithm for $[1, j]$-Dominating Set on split graphs is optimal under the Strong Exponential Time Hypothesis (SETH).
- Assuming SETH, we provide a lower bound for the running time of any algorithm solving the $[1, 2]$-Total Dominating Set problem parameterized by pathwidth.
- Finally, we study another variant of Dominating Set, called Restrained Dominating Set, that asks if there is a dominating set $D$ of $G$ of size at most $k$ such that no vertex outside of $D$ has all of its neighbors in $D$. We prove this variant is W[1]-hard even on 3-degenerate graphs.



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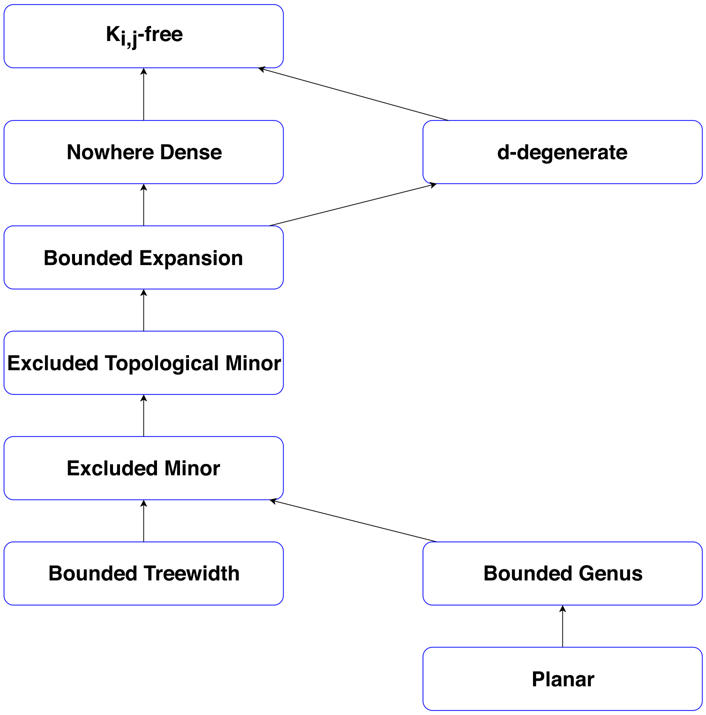


**Fig. 1.** Inclusion relations among well-studied classes of sparse graphs [12].

## 1. Introduction

A dominating set of a graph $G$ is a subset $D$ of vertices such that each vertex in $V(G) \setminus D$ is adjacent to at least one vertex in $D$. Various extensions of domination, such as independent, total, efficient, and perfect domination, have been introduced and widely studied both combinatorially and algorithmically. A discussion of these extensions can be found in [1]. A $[1, j]$-dominating set, as first defined in [2], is a set that dominates the vertices of the graph but every vertex outside of the set must have at most $j$ neighbors in it. In [2], it was shown that a minimum $[1, 2]$-dominating set and a minimum dominating set are of the same size in several classes of graphs such as claw-free graphs, $P_4$-free graphs, and caterpillars. It was also shown that the problem is NP-complete, even for bipartite graphs. In the $[1, j]$-Dominating Set problem the input is a graph $G$ and a positive integer $k$. The objective is to test whether there is a $[1, j]$-dominating set of size at most $k$. The authors in [2] raised several open problems, including whether restricting to specific classes of graphs leads to strictly better upper bounds for the size of $[1, j]$-dominating sets and whether $[1, j]$-Dominating Set is efficiently solvable on trees. In [3] the first question was answered negatively for the classes of planar, bipartite, and triangle-free graphs in which the smallest $[1, 2]$-dominating set is the entire set of vertices. In [4] the second question was answered positively via a linear-time algorithm. In [5], the $[1, j]$-Dominating Set problem was shown to be NP-hard even for chordal and planar graphs. However, for a constant $j$, a polynomial-time algorithm running in time $O(n^j p(\lg n))$ where $p$ is a polynomial function, was obtained for $n$-vertex split graphs [5]. This is in contrast to the classic Dominating Set problem which is NP-hard for this class of graphs.

The Dominating Set problem has been widely studied in the realm of parameterized complexity. In general, finding a dominating set of size $k$ is a canonical W[2]-complete problem and therefore, unlikely to admit an FPT algorithm [6]. Moreover, the problem remains W[2]-complete for split and bipartite graphs [7]. Nevertheless, there are interesting classes of sparse graphs for which the Dominating Set problem admits FPT algorithms. For example, there is an $O^*(3^{\text{tw}})$-time[1] algorithm for graphs of treewidth at most tw [8,9], and FPT algorithms for nowhere dense graphs [10] and $d$-degenerate graphs [11]. Also, an FPT algorithm was reported in [12] for $t$-biclique-free graphs, i.e., graphs that do not contain $K_{t,t}$ as a subgraph. To the best of our knowledge, this is the largest class of graphs for which the Dominating Set problem is known to be fixed-parameter tractable; $d$-degenerate and nowhere dense graphs are subclasses of $t$-biclique-free graphs. Fig. 1 shows an inclusion hierarchy among the most commonly studied classes of sparse graphs.

Another variant of dominating sets and $[1, j]$-dominating sets is $[1, j]$-total dominating sets. For a graph $G$, a subset $D \subseteq V(G)$ is called a $[1, j]$-total dominating set if $1 \le |N(v) \cap D| \le j$ for all $v \in V(G)$, where $N(v)$ denotes the open neighborhood of $v$ in $G$. In the $[1, j]$-Total Dominating Set problem we are given a graph $G$ and a positive integer $k$.

[1] We sometimes use the modified big-Oh notation $O^*$ that suppresses all factors bounded polynomially in the input size.

The objective is to check whether $G$ admits a $[1, j]$-total dominating set of size at most $k$. $[1,2]$-Total Dominating Set is NP-complete even for bipartite graphs [13]. Sharp upper bounds on the $[1,2]$-total domination number of a graph are investigated in [3,14]. Using a result of Rooij et al. [8], we can show that $[1, j]$-Dominating Set and $[1, j]$-Total Dominating Set are solvable in time $O^*((j+2)^{\mathrm{tw}})$ and $O^*((2j+2)^{\mathrm{tw}})$, respectively, on graphs of treewidth at most tw. A third variant of domination, called *restrained domination* was introduced in [15]. In the Restrained Dominating Set problem, the input is a graph $G$ and a positive integer $k$. The objective is to test whether there is a dominating set $D$ of size at most $k$ such that for any $v \in V(G) \setminus D$, $v$ is adjacent to a vertex in $V(G) \setminus D$. In other words, a set $D \subseteq V$ of a graph $G$ is called a *restrained dominating set* if every vertex not in $D$ is adjacent to a vertex in $D$ and a vertex in $V \setminus D$.

**Our Contribution.** In this paper, we study $[1, j]$-Dominating Set, $[1, j]$-Total Dominating Set, and Restrained Dominating Set from the parameterized complexity perspective. We prove the following results.

- For any $\epsilon > 0$, there is no algorithm with running time $O(n^{j-\epsilon})$ for $[1, j]$-Dominating Set on split graphs assuming the Strong Exponential Time Hypothesis (SETH).
- For the case of $d$-degenerate graphs, we tighten the complexity results and show that $[1, j]$-Dominating Set is W[1]-hard for $d = j+1$. As $[1, j]$-Dominating Set generalizes Dominating Set, the class of degenerate graphs is one example where the latter problem is "easy" (under the parameterized complexity framework) while the former (generalized) problem is hard. This result also implies that $[1, j]$-Dominating Set is W[1]-hard on classes of biclique-free graphs, the largest class of graphs for which the Dominating Set problem is known to be fixed-parameter tractable.
- $[1, j]$-Dominating Set is FPT on classes of nowhere dense graphs.
- There is no algorithm for $[1,2]$-Total Dominating Set running in time $O^*((4-\epsilon)^{\mathrm{pw}})$ assuming SETH, where pw is the pathwidth of the input graph.
- Restrained Dominating Set is W[1]-hard even on 3-degenerate graphs.

We begin by defining some basic terminology and notation in Section 2. Then the next sections each address one of the results mentioned above.

## 2. Preliminaries

**Graphs.** We assume $G$ is a simple graph with vertex set $V(G)$ and edge set $E(G)$. For brevity, we often denote these sets by $V$ and $E$. We let $n = |V(G)|$ denote the order of $G$. For a vertex $v \in V$, the open neighborhood of $v$, denoted by $N(v)$, is defined as $\{u : \{u, v\} \in E\}$ and the closed neighborhood $N[v]$ is defined as $N(v) \cup \{v\}$. For a set $S \subseteq V$, we use $N(S)$ and $N[S]$ to denote the open and closed neighborhood of $S$, respectively. That is, $N[S] = \bigcup_{v \in S} N[v]$ and $N(S) = N[S] \setminus S$. For a set $U \subseteq V$, we use $G[U]$ to denote the subgraph of $G$ induced on $U$.

A tree decomposition of a graph $G$ is a tree $T$ in which each vertex $x \in T$ has an assigned set of vertices $B_x \subseteq V(G)$ (called a bag) which satisfies the following properties:

(i) $\bigcup_{x \in T} B_x = V(G)$;
(ii) For any $\{u, v\} \in E$, there exists $x \in V(T)$ such that $u, v \in B_x$;
(iii) For any $v \in V(G)$, the subtree of $T$ induced on $\{x \in V(T) : v \in B_x\}$ is connected.

The *width* of a tree decomposition $T$ is $\max_{x \in V(T)}(|B_x| - 1)$. The treewidth of $G$, denoted by $tw(G)$, is the minimum width over all tree decompositions of $G$. The pathwidth of $G$, denoted by $pw(G)$, is the minimum width over all tree decompositions $T$ of $G$, where $T$ is a path.

**Parameterized complexity.** We now review some necessary concepts from parameterized complexity. For more details, we refer the reader to [16,17]. Given a finite alphabet $\Sigma$, a parameterization of $\Sigma^*$ is a function $p : \Sigma^* \to \mathbb{N}$. A parameterized language $L$ is a subset of $\{(x, k) \mid x \in \Sigma^* \wedge k = p(x)\}$. Here $k$ is called the parameter. A parameterized language $L \subseteq \Sigma^* \times \mathbb{N}$ is called fixed-parameter tractable (FPT) if there exists an algorithm $A$ (called an *FPT algorithm*) and a computable function $f : \mathbb{N} \to \mathbb{N}$ such that given $(x, k) \in \Sigma^* \times \mathbb{N}$, the algorithm $A$ correctly decides whether $(x, k) \in L$ in time $f(k) \cdot |(x, k)|^{O(1)}$. The class of all fixed-parameter tractable problems is denoted by FPT.

**Definition 2.1.** Let $L$ and $L'$ be two parameterized languages with parameterization functions $p$ and $p'$, respectively. An FPT reduction from $L$ to $L'$ is a mapping $\rho : \Sigma^* \to \Sigma^*$ such that the following holds.

- For all $x \in \Sigma^*$, $(x, p(x)) \in L$ if and only if $(\rho(x), p'(\rho(x))) \in L'$.
- There exists a computable function $g : \mathbb{N} \to \mathbb{N}$ such that for all $x \in \Sigma^*$, $p'(\rho(x)) \leq g(p(x))$.
- $\rho$ is computable in FPT time, i.e., there exists a computable function $f$ such that $\rho(x)$ is computable in time $O(f(p(x)) \cdot |x|^{O(1)})$.

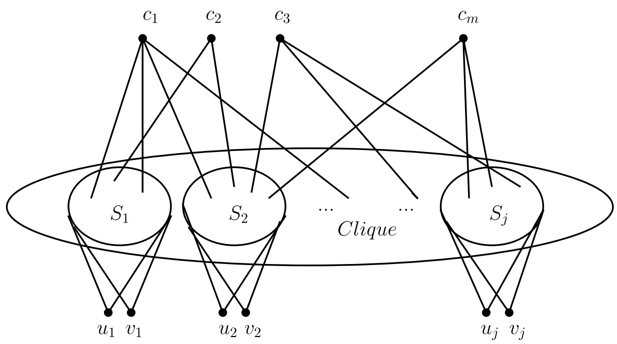


**Fig. 2.** Constructed Split Graph.

To classify problems that are not FPT, Downey and Fellows [17] introduced the W-hierarchy. The hierarchy consists of complexity class W[$t$] for every integer $t \in \mathbb{N}$ such that $W[t] \subseteq W[t+1]$, for all $t$. More generally, FPT $\subseteq$ W[1] $\subseteq$ W[2] $\subseteq \cdots \subseteq$ W[t]. For our purposes, it is sufficient to note that these classes are closed under FPT reductions.

**ETH and SETH.** For $q \geq 3$, let $\delta_q$ be the infimum of the set of constants $c$ for which there exists an algorithm solving $q$-SAT with $n$ variables and $m$ clauses in time $2^{cn} \cdot m^{O(1)}$. The *Exponential-Time Hypothesis (ETH)* and the *Strong Exponential-Time Hypothesis (SETH)* are then formally defined as follows. ETH conjectures that $\delta_3 > 0$ and SETH conjectures that $\lim_{q \to \infty} \delta_q = 1$. In other words, SETH conjectures that for all $0 < \epsilon < 1$, there exists a (large) $q = q(\epsilon)$ such that $q$-SAT cannot be solved in time $O(2^{(1-\epsilon)n})$, where $n$ is the number of variables in the input formula.

## 3. Split graphs

A graph $G$ is called a split graph if its vertices can be partitioned into $V_1 \uplus V_2$ such that $G[V_1]$ is a complete graph and $G[V_2]$ is an empty graph (i.e., $V_2$ is an independent set in $G$). Several domination-like problems such as domination, total domination, and $k$-tuple domination are known to be NP-complete even on split graphs. On the other hand, there is an $n^j \cdot (\lg n)^{O(1)}$-time algorithm for $[1, j]$-Dominating Set on split graphs [5]. In this section, we prove that this is optimal, in the sense that one cannot obtain an $O(n^{j-\epsilon})$ algorithm for $[1, j]$-Dominating Set on split graphs unless SETH fails.

**The reduction.** Given an instance $I = C_1 \wedge C_2 \wedge \ldots \wedge C_m$ of $q$-SAT over the set $X = \{x_1, \ldots, x_n\}$ of variables, we construct a graph $G_I$ as follows.

- We partition the set $X$ into $j$ subsets $X_1, \ldots, X_j$ of size at most $\lceil n/j \rceil$.
- For each $X_i$ we add a set $S_i$ of $2^{|X_i|}$ vertices to the graph, each corresponding to one possible valuation of the variables in $X_i$. We also add two distinguished vertices $u_i$ and $v_i$ and connect them to all of $S_i$.
- We connect all the vertices in $\bigcup_{i=1}^{j} S_i$. That is $\bigcup_{i=1}^{j} S_i$ forms a clique in $G_I$.
- For every clause $C_i$, we add a vertex $c_i$ and connect it to every vertex $w \in S_j$ where the valuation corresponding to $w$ satisfies $C_i$.

This completes the construction of $G_I$. See Fig. 2 for an illustration. Our reduction algorithm will output the instance $(G_I, j)$ of $[1, j]$-Dominating Set. We now proceed to proving the correctness of the reduction.

**Remark 3.1.** The graph $G_I$ is a split graph.

**Lemma 3.2.** *If $I$ is satisfiable then $G_I$ has a $[1, j]$-dominating set of size $j$.*

**Proof.** Take one satisfying valuation of $I$ and let $s_1 \in S_1, s_2 \in S_2, \ldots, s_j \in S_j$ be the vertices of $G_I$ that correspond to this valuation. We claim that $S = \{s_1, s_2, \ldots, s_j\}$ is a $[1, j]$-dominating set. Every vertex in any of the $S_i$'s is dominated by all of $S$, every $\{u_i, v_i\}$ pair is dominated only by the corresponding $s_i$ and every $c_i$ is dominated by a non-empty subset of $S$, i.e., the vertices whose corresponding valuation forces satisfaction of $C_i$. Given that $|S| = j$, there can be at most $j$ such vertices. $\square$

**Lemma 3.3.** *If $G_I$ has a $[1, j]$-dominating set of size $j$ then $I$ is satisfiable.*

**Proof.** First note that any dominating set of $G_I$ of size $j$ is also a $[1, j]$-dominating set. Let $S$ be a dominating set of size at most $j$ (and hence a $[1, j]$-dominating set) in $G_I$. Since $\{N[u_i]\colon i \in \{1, \ldots, j\}\}$ are pairwise vertex disjoint we have that $|S| = j$ and $|S \cap N[u_i]| = 1$ for all $i \in \{1, \ldots, j\}$. Moreover, since $N(u_i) = N(v_i)$, we conclude that $S \subseteq (\bigcup_{i=1}^{j} S_i)$. Consider

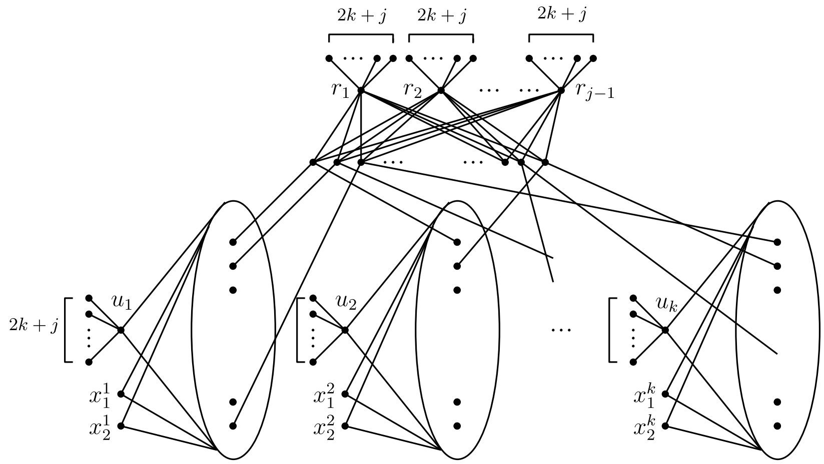


**Fig. 3.** Reducing the MULTICOLORED INDEPENDENT SET problem to the $[1, j]$-DOMINATING SET problem.

the valuation of variables in $X$ that corresponds to the vertices in $S$. This valuation satisfies every $C_i$, because $S$ dominates every $c_i$. $\square$

**Theorem 3.4.** *For any $\epsilon < 1$ and constant $j$, there is no $O(n^{j-\epsilon})$ time algorithm for* $[1, j]$-DOMINATING SET *on split graphs unless SETH fails.*

**Proof.** For the sake of contradiction assume that there is an $O(n^{j-\epsilon})$ time algorithm $\mathcal{A}$ for $[1, j]$-DOMINATING SET on split graphs. Then we claim that SETH is false. Let $\epsilon' = (1 - \frac{\epsilon}{j})$. Let $q = q(\epsilon')$ be the constant defined in the SETH conjecture. Given a $q$-SAT instance $I$ we construct the instance $(G_I, j)$ as mentioned in the reduction. By Lemmas 3.2 and 3.3 we know that $I$ is satisfiable if and only if $(G_I, j)$ is a yes-instance. Therefore, we use algorithm $\mathcal{A}$ to solve $q$-SAT. Now consider the running time for solving $q$-SAT using $\mathcal{A}$. The construction of $(G_I, j)$ takes time $2^{\lceil n/j \rceil} \cdot n^{O(1)}$ and the number of vertices in $G_I$ is at most $O(2j+m+j2^{\lceil n/j \rceil}) = O(2^{\lceil n/j \rceil} \cdot n^{O(1)})$. Thus the algorithm $\mathcal{A}$ on instance $(G_I, j)$ takes time $O(2^{\lceil n/j \rceil (j-\epsilon)} \cdot n^{O(1)}) = O(2^{(n/j+(j-1))(j-\epsilon)} \cdot n^{O(1)}) = O(2^{(n/j)(j-\epsilon)} \cdot n^{O(1)}) = O(2^{n(1-\epsilon/j)} \cdot n^{O(1)}) = O(2^{\epsilon' n} \cdot n^{O(1)})$. This refutes SETH and the proof of the theorem is complete. $\square$

## 4. Degenerate graphs

A graph G is $d$-degenerate if every subgraph of $G$ contains a vertex of degree at most $d$. Equivalently, a graph $G$ is $d$-degenerate if and only if there exists an elimination ordering on its vertices such that every vertex has at most $d$ neighbors appearing later in the ordering. In this section we prove the following result.

**Theorem 4.1.** $[1, j]$-DOMINATING SET *parameterized by solution size is W[1]-hard on graphs of degeneracy $j+1$.*

The following parameterized problem, proved to be W[1]-hard in [18], is used in our proof. In the MULTICOLORED INDEPENDENT SET problem, we are given a graph $G$ and a proper vertex coloring of $V(G)$ with $k$ colors. The parameter $k$ is equal to the number of colors and the goal is to find a $k$-sized independent set in $G$ containing exactly one vertex from each color class (such independents sets are called *k-colored independent sets*). We note that the coloring of $G$ need not be proper but the given definition of the problem is more suitable for our argumentation. We shall reduce the MULTICOLORED INDEPENDENT SET problem (with parameter $k$) to the problem of finding a $[1, j]$-dominating set of size at most $2k + j - 1$ in a graph of degeneracy $j+1$.

**The reduction.** Let $k$ be an integer and $G$ be a proper $k$-vertex colored graph such that its vertices are partitioned into $k$ groups $V_1, V_2, \ldots, V_k$, where each group corresponds to an independent set of the same color. Now we construct an instance $(G', 2k+j-1)$ of $[1, j]$-DOMINATING SET as follows. For every edge $e = \{u, v\} \in E(G)$, we replace it by a path $uv_ev$, where $v_e$ is a new vertex corresponding to the edge $e$. Let $S_E = \{v_e \ : \ e \in E(G)\}$. For each group $V_i$, $1 \le i \le k$, we build a $K_{1,2k+j}$ graph centered at a new vertex $u_i$. We also add new vertices $x_1^i$ and $x_2^i$ and connect the vertices $u_i, x_1^i$ and $x_2^i$ to the vertices of $V_i$. Moreover, we build $j-1$ star graphs $K_{1,2k+j}$ centered at vertices $r_1, \ldots, r_{j-1}$ and make $r_1, \ldots, r_{j-1}$ adjacent to the vertices in $S_E$. This concludes the construction of $G'$. See Fig. 3 for an illustration. Now we output $(G', 2k+j-1)$ as an instance of $[1, j]$-DOMINATING SET. Clearly our reduction takes time polynomial in $|V(G)|$ and $k$.

**Lemma 4.2.** *The constructed graph $G'$ has degeneracy $j+1$.*

**Proof.** The proof is by constructing a degeneracy ordering, which is an ordering on the vertices that we get from repeatedly removing a vertex of minimum degree in the remaining subgraph. First, we put all of the degree-one vertices in the degeneracy ordering and delete them. In the remaining subgraph, we select all the vertices in $S_E$ and put them in ordering, because every vertex in $S_E$ has degree $j+1$ in $G'$. After removing all vertices of $S_E$ from the graph, each vertex in any block $V_i$, for $1 \leq i \leq k$, has degree three because after removing $S_E$ such vertices are only connected to $x_1^i, x_2^i$, and $u_i$. So, next we can put the vertices $\bigcup_{i=1}^k V_i$ in the ordering. Finally we add all the remaining vertices. □

Lemmas 4.2, 4.3, and 4.4 below, imply Theorem 4.1.

**Lemma 4.3.** *If there exists a k-colored independent set in G then there exists a* $[1, j]$*-dominating set of size* $2k+j-1$ *in* $G'$.

**Proof.** Suppose that $S$ is a $k$-colored independent set in $G$. We claim that $D = S \cup \{u_i : i \in \{1,\dots,k\}\} \cup \{r_i : i \in \{1,\dots,j-1\}\}$ is a $[1, j]$-dominating set of $G'$. Clearly, the size of $D$ is $2k+j-1$. Each vertex $v_j \in V_i$ is dominated only by $u_i$ in $D$. Moreover, each pair of $x_1^i, x_2^i$ vertices is dominated by the single vertex in $V_i \cap S$. All the vertices in $S_E$ are dominated by $\{r_i : i \in \{1,\dots,j-1\}$ and by at most one vertex from $S$ (since $S$ is an independent set). Moreover all the degree one vertices in $G'$ are dominated exactly once by $D$. □

**Lemma 4.4.** *If there exists a* $[1, j]$*-dominating set of size* $2k+j-1$ *in* $G'$ *then there exists a k-colored independent set in G.*

**Proof.** Let $D$ be a $[1, j]$-dominating set of size at most $2k+j-1$. First, note that since $|D| \leq 2k+j-1$, we have that $\{r_1,\dots,r_{j-1}\} \cup \{u_i,\dots,u_k\} \subseteq D$ (because each $u_i$ and $r_i$ is connected to $2k+j$ degree one vertices). We claim that $S = D \setminus (\{r_1,\dots,r_{j-1}\} \cup \{u_i,\dots,u_k\})$ is a $k$-colored independent set in $G$. Clearly $|S| \leq k$. Also, to dominate all the vertices $x_1^i, x_i^2$ for $1 \leq i \leq k$, we should have exactly one vertex from each $V_i$. Therefore $S \subseteq V(G)$ and $|S \cap V_i| = 1$ for all $1 \leq i \leq k$. Suppose $u, v \in S$ is adjacent in $G$. Then the vertex $v_e$, where $e = \{u, v\}$ is dominated $j+1$ times by $D$ (because $v_e$ is adjacent to $\{u, v\}$ and $\{r_1,\dots,r_{j-1}\}$). Therefore, since $D$ is a $[1, j]$-dominating set in $G'$, $S$ is a $k$-colored independent set in $G$ This completes the proof of the lemma. □

## 5. Nowhere dense graphs

The notion of nowhere denseness was introduced by Nešetřil and Ossona de Mendez [19,20] as a general model of uniform sparseness of graphs. Many familiar classes of sparse graphs, like planar graphs, graphs of bounded tree-width, graphs of bounded degree, and all classes that exclude a fixed (topological) minor, are nowhere dense. An important and related concept is the notion of a graph class of bounded expansion, which was also introduced by Nešetřil and Ossona de Mendez [21–23].

**Definition 5.1.** Let $H$ be a graph and let $r \in \mathbb{N}$. An $r$-subdivision of $H$ is obtained by replacing all edges of $H$ by internally vertex disjoint paths of length at most $r$.

**Definition 5.2.** A class $\mathcal{C}$ of graphs is nowhere dense if there exists a function $t\colon \mathbb{N} \to \mathbb{N}$ such that for all $q \in \mathbb{N}$ and for all $G \in \mathcal{C}$, we do not find an $q$-subdivision of the complete graph $K_{t(q)}$ as a subgraph of $G$. Otherwise, $\mathcal{C}$ is called somewhere dense.

**Definition 5.3.** A class $\mathcal{C}$ of graphs has bounded expansion if there exists a function $d\colon \mathbb{N} \to \mathbb{N}$ such that for all $r \in \mathbb{N}$ and all graphs $H$, where an $r$-subdivision of $H$ is a subgraph of $G$ for some $G \in \mathcal{C}$, satisfy $|E(H)|/|V(H)| \leq d(r)$.

Every class of bounded expansion is nowhere dense, which in turn excludes some biclique as a subgraph and hence is biclique-free. For an extensive background on bounded expansion and nowhere dense graphs we refer to the textbook of Nešetřil and Ossona de Mendez [24].

Before we state our result, we quickly recall the necessary definitions from logic. For our purpose, it suffices to consider first-order logic over the vocabulary of graphs. We refer to the textbook [25] for an extensive background on logic. A (relational) vocabulary is a finite set of relation symbols, each with a prescribed arity. Let $\sigma$ be a vocabulary. A $\sigma$-structure $A$ consists of a (not necessarily finite) set $V(A)$, called the universe or the vertex set of $A$, and for each $k$-ary relation symbol $R \in \sigma$ a $k$-ary relation $R(A) \subseteq V(A)^k$. A structure $A$ is finite if its universe is finite. For example, graphs may be viewed as $\{E\}$-structures, where the vertex set of the graph is the universe and $E$ is a binary relation symbol. First-order formulas of vocabulary $\sigma$ are formed from atomic formulas $x = y$ and $R(x_1,\dots,x_k)$, where $R \in \sigma$ is a $k$-ary relation symbol and $x, y, x_1,\dots,x_k$ are variables (we assume that we have an infinite supply of variables) by the usual Boolean connectives $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), and existential and universal quantifications $\exists x$ and $\forall x$, respectively.

The set of all first-order formulas of vocabulary $\sigma$ is denoted by FO[$\sigma$], and the set of all first-order formulas by FO. The free variables of a formula are those not in the scope of a quantifier, and we write $\phi(x_1,\dots,x_k)$ to indicate that the free variables of the formula $\phi$ are $x_1,\dots,x_k$. A sentence is a formula without free variables. To define the semantics, we inductively define a satisfaction relation $\models$, where for a $\sigma$-structure $A$, a formula $\phi(x_1,\dots,x_k)$, and elements $a_1,\dots,a_k \in V(A)$, $A \models \phi(a_1,\dots,a_k)$ means that $A$ satisfies $\phi$ if the free variables $x_1,\dots,x_k$ are interpreted by $a_1,\dots,a_k$, respectively. If $\phi(x_1,\dots,x_k) = R(x_1,\dots,x_k)$ is atomic, then $A \models \phi(a_1,\dots,a_k)$ if $(a_1,\dots,a_k) \in R(A)$. The meaning of the equality symbol, the Boolean connectives, and the quantifiers is the usual one. For example, consider the formula $\phi(x_1,x_2) = \forall y(x_1 = y \vee x_2 = y \vee E(x_1,y) \vee E(x_2,y))$ in the vocabulary $\{E\}$ of graphs. For every graph $G$ and vertices $v_1,v_2 \in V(G)$ we have $G \models \phi(v_1,v_2)$ if any only if $\{v_1,v_2\}$ is a dominating set of $G$. Thus $G$ satisfies the sentence $\exists x_1 \exists x_2 \phi(x_1,x_2)$ if and only if it has a (nonempty) dominating set of size at most 2.

**Theorem 5.4.** *The* $[1, j]$-Dominating Set *problem parameterized by solution size $k$ is fixed-parameter tractable on nowhere dense classes of graphs.*

**Proof.** Our proof is based on a result of Grohe, Kreutzer, and Siebertz [26], which states that for every first-order sentence $\psi$ (or formula without free variables), every nowhere dense class $\mathcal{C}$ of graphs and every real $\epsilon > 0$, there exists a constant $f(|\psi|,\epsilon)$, such that given an $n$-vertex graph $G \in \mathcal{C}$, one can decide in time $f(|\psi|,\epsilon)\cdot n^{1+\epsilon}$ whether $\psi$ holds in $G$.

It is easy to verify that the $[1, j]$-Dominating Set problem is expressible in FO. Let $\psi$ be the following sentence.

$$\exists v_1, v_2, \dots, v_k \forall u \Big( (u = v_1 \vee \dots \vee v_k) \vee ((\phi_1(u, v_1, v_2, \dots, v_k) \vee \phi_2(u, v_1, v_2, \dots, v_k)$$
$$\dots \vee \phi_j(u, v_1, v_2, \dots, v_k)) \Big),$$

where the function $\phi_i(u, v_1, v_2, \dots, v_k)$ is true where the vertex $u$ is adjacent to exactly $i$ vertices of $v_1, v_2, \dots, v_k$, which can be represented using a formula of length bounded by a function of $i$ and $k$. Note that the length of $\psi$ is bounded by a function depending only on $k$ (and on $j$, though only as a fixed constant). Then, by fixing any $\epsilon > 0$ and using the result of [26], we conclude that $[1, j]$-Dominating Set is fixed-parameter tractable parameterized by solution size $k$ on every nowhere dense class $\mathcal{C}$. □

## 6. Bounded treewidth graphs

It is well-known [16] that the Dominating Set problem can be solved in $O^*(3^t)$ time on graphs of treewidth at most $t$. Lokshtanov et al. [9] showed that this is essentialy optimal, i.e., they showed that the problem cannot be solved in $O^*((3-\epsilon)^t)$ time unless SETH fails. The situation is almost identical for the Connected Dominating Set problem. The problem can be solved in $O^*(4^t)$ time, but cannot be solved in time $O^*(4-\epsilon)^t$ unless SETH fails [27]. The $[1, j]$-Dominating Set problem is a special case of the $(\sigma,\rho)$-domination problem. The concept of $(\sigma,\rho)$-domination was introduced by Telle [28]. Let $\sigma,\rho$ be a pair of non-empty sets of non-negative integers. A set $S$ of vertices of a graph $G$ is called $(\sigma,\rho)$-dominating if for every vertex $v \in S$, $|S \cap N(v)| \in \sigma$, and for every $v \notin S$, $|S \cap N(v)| \in \rho$. $[1, j]$-Dominating Set and $[1, j]$-Total Dominating Set are special cases of the $(\sigma,\rho)$-Dominating Set problem. For $[1, j]$-Dominating Set, we set $\sigma = \{0, 1, \dots\}$ and $\rho = \{1,\dots, j\}$. On the other hand for $[1, j]$-Total Dominating Set we set $\sigma = \rho = \{1,\dots, j\}$. The next result is due to the work of Rooij et al. [8].

**Proposition 6.1** *([8]).* $[1, j]$-Dominating Set *and* $[1, j]$-Total Dominating Set *are solvable in time* $O^*((j+2)^{\mathrm{tw}})$ *and* $O^*((2j+2)^{\mathrm{tw}})$, *respectively, on graphs of treewidth at most* tw.

We prove that the $[1, 2]$-Total Dominating Set problem cannot be solved in time $O^*(4-\epsilon)^{\mathrm{pw}}$ (unless SETH fails), which is also a lower bound in terms of the treewidth of the input graph. It remains open whether a similar result holds for $[1, 2]$-Dominating Set. Our proof closely follows the work of Cygan et al. [27] and we use the notions of path decompositions, pathwidth, tree decompositions, treewidth, and mixed search games. We give a reduction from CNF-SAT to the $[1, 2]$-Total Dominating Set problem and prove that the reduced graph has pathwidth at most $\frac{n}{2} + O(1)$, where $n$ is the number of variables in the input CNF-SAT formula.

**The reduction.** Given $\epsilon > 0$ and an instance $\Phi$ of CNF-SAT with $n$ variable and $m$ clauses, we construct a graph $G$ as follows. We assume that the number of variables $n$ is even, otherwise we add a single dummy variable. We partition the variables of $\Phi$ into groups $F_1, F_2, \dots, F_{n'}$, each of size two, where $n' = n/2$. We let $a = m(n+1)$.

For each $1 \le t \le n'$ we create a path $P_t$ of length $4a = 4m(n+1)$ consisting of vertices $v^\alpha_{t,q}$ and $h^\alpha_{t,q}$, where $1 \le \alpha \le 2$ and $0 \le q < a$. The vertices are arranged on the path in the following order: $v^1_{t,0}, h^1_{t,0}, v^2_{t,0}, h^2_{t,0}, v^1_{t,1}, h^1_{t,1}, v^2_{t,1}, h^2_{t,1}, \dots, v^1_{t,a-1}, h^1_{t,a-1}, v^2_{t,a-1}, h^2_{t,a-1}$. We let $\mathcal{V}$ and $\mathcal{H}$ denote the sets of all $v^\alpha_{t,q}$ vertices and $h^\alpha_{t,q}$ vertices, respectively.

For each vertex $v^\alpha_{t,q}$, we add a forcing gadget consisting of a 4-cycle with one additional pendant vertex connected to a vertex which we denote as the root vertex $r^\alpha_{t,q}$. We add an edge between $v^\alpha_{t,q}$ and the root of the cycle $r^\alpha_{t,q}$. Similarly, for

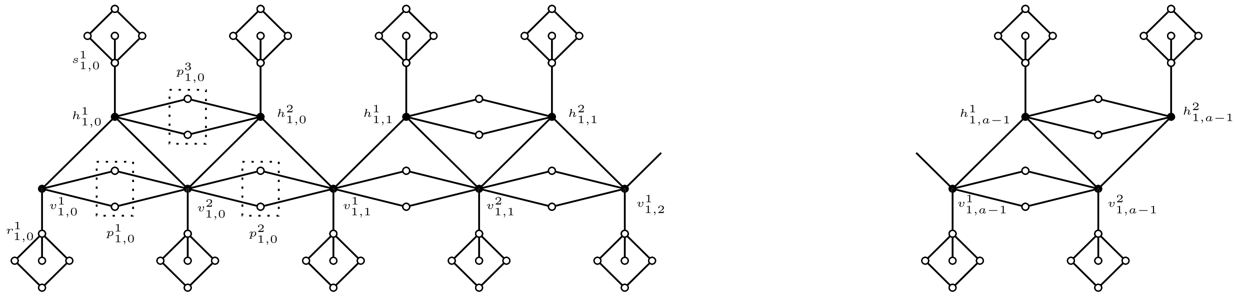


**Fig. 4.** Parts of the construction.

each vertex $h^\alpha_{t,q}$, we add a forcing gadget consisting of a 4-cycle rooted at vertex $s^\alpha_{t,q}$ (i.e. the pendant vertex is connected to $s^\alpha_{t,q}$). We add an edge between $h^\alpha_{t,q}$ and $s^\alpha_{t,q}$ (see Fig. 4). Note that any dominating set in the graph will contain at least two vertices from a forcing gadget and if a $[1,2]$-total dominating set contains only two vertices from a forcing gadget, then it includes the root vertex and one of its neighbors on the 4-cycle (by the definition of a $[1,2]$-total dominating set).

Next, we add three pairs of guard vertices $p^1_{t,q}$, $p^2_{t,q}$, and $p^3_{t,q}$, for each $1 \le t \le n'$ and $0 \le q < a$. Each of the vertices in these pairs are of degree two and are connected to other vertices as follows: (i) vertices in $p^1_{t,q}$ are adjacent to $v^1_{t,q}$ and $v^2_{t,q}$; (ii) vertices in $p^2_{t,q}$ are adjacent to $v^2_{t,q}$ and $v^2_{t,q+1}$; and (iii) vertices in $p^3_{t,q}$ are adjacent to $h^1_{t,q}$ and $h^2_{t,q}$. This structure forces a dominating set to either contain the vertices from the guard sets or one of their two neighbors. For instance, to dominate the pair $p^1_{t,q}$, either both vertices in $p^1_{t,q}$ must be in the dominating set or one of $v^1_{t,q}$ or $v^2_{t,q}$. We use $\mathcal{G}$ to denote the set of all guard vertices The intuition of the construction made so far is as follows. For each two-variable block $F_t$ we encode any assignment of the variables in $F_t$ as a choice of whether to take $v^1_{t,q}$ or $v^2_{t,q}$ and $h^1_{t,q}$ or $h^2_{t,q}$ into the dominating set. This concludes the construction of the "variable gadgets" which are required for encoding an assignment. The forcing gadgets attached to vertices in $\mathcal{V}$ and $\mathcal{H}$ will guarantee that those vertices are all dominated at least once.

We now add "clause gadgets" required for checking the satisfiability of $\Phi$. For each clause $C_i$, we build $(n+1)$ vertices $c_{i,j}$, one for each $0 \le j \le n$. Consider a clause $C_i$ and a group of variables $F_t = \{x^1_t, x^2_t\}$. If $x^1_t$ occurs positively as the $\ell$th literal in $C_i$, then connect $v^1_{t,mj+i}$ to $c_{i,j}$ via a path of length five by adding four vertices $w^\ell_{i,j}$, $x^\ell_{i,j}$, $y^\ell_{i,j}$ and $z^\ell_{i,j}$ and edges $\{c_{i,j}, w^\ell_{i,j}\}$, $\{w^\ell_{i,j}, x^\ell_{i,j}\}$, $\{x^\ell_{i,j}, y^\ell_{i,j}\}$, $\{y^\ell_{i,j}, z^\ell_{i,j}\}$, and $\{z^\ell_{i,j}, v^1_{t,mj+i}\}$. If $x^1_t$ occurs negatively as the $\ell$th literal in $C_i$, we connect $v^2_{t,mj+i}$ to $c_{i,j}$ again via a path of length five by adding four vertices $w^\ell_{i,j}$, $x^\ell_{i,j}$, $y^\ell_{i,j}$ and $z^\ell_{i,j}$ and edges $\{c_{i,j}, w^\ell_{i,j}\}$, $\{w^\ell_{i,j}, x^\ell_{i,j}\}$, $\{x^\ell_{i,j}, y^\ell_{i,j}\}$, $\{y^\ell_{i,j}, z^\ell_{i,j}\}$, and $\{z^\ell_{i,j}, v^2_{t,mj+i}\}$. Similarly, if $x^2_t$ occurs positively as the $\ell$th literal in $C_i$, we connect $h^1_{t,mj+i}$ to $c_{i,j}$ via a path of length five and if it occurs negatively, we connect $h^2_{t,mj+i}$ to $c_{i,j}$ via a path of length five. Let $\mathcal{W}$, $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ denote the sets of all $w$, $x$, $y$, and $z$ vertices added between clause vertices and vertices in $\mathcal{V} \cup \mathcal{H}$, respectively. This concludes the construction of the graph $G$ (see Fig. 5).

**Lemma 6.2.** *If $\Phi$ is satisfiable then $G$ has a $[1,2]$-total dominating set $D$ of size $k = 10an' + 2(n+1)\Sigma_{i=1}^m |C_i|$.*

**Proof.** Consider a satisfying assignment $\phi$ for $\Phi$. We construct a $[1,2]$-total dominating set $D$ of size $k$ in $G$ as follows. We first add the root vertex and one of its neighbors on the 4-cycle of each forcing gadget to $D$. Then, for each block $F_t = \{x^1_t, x^2_t\}$ and each $0 \le q < a$, we add the vertex $v^1_{t,q}$ to $D$ if the value of $x^1_t$ is true, otherwise we add $v^2_{t,q}$ to $D$. Similarly, if the value of $x^2_t$ is true, we add $h^1_{t,q}$ to $D$, otherwise we add $h^2_{t,q}$. Since the clause $C_i$ is satisfied, at least one of the vertices $v^\alpha_{t,q}$ or $h^\alpha_{t,q}$ (for some $t, \alpha$) will be in the dominating set $D$. Hence, the corresponding $z^\ell_{i,j}$ vertex will be dominated, for some $1 \le \ell \le |C_i|$. We can therefore add vertices $x^\ell_{i,j}$ and $w^\ell_{i,j}$ to $D$ as to dominate $c_{i,j}$ and $y^\ell_{i,j}$ (and maintain the $[1,2]$-total dominating set property). To dominate the remaining vertices in the clause gadget for $c_{i,j}$, we add all $y^{\ell'}_{i,j}$ and $x^{\ell'}_{i,j}$ vertices, where $1 \le \ell' \le |C_i|$ and $\ell' \ne \ell$.

It is clear that for each clause $C_i$, we add $2(n+1)|C_i|$ vertices to $D$, for a total of $2(n+1)\Sigma_{i=1}^m |C_i|$ vertices. Moreover, for each path $P_t$, we add $10a$ vertices to $D$. Therefore, accounting for the fact that we have $n'$ paths in total, the total number of vertices in $D$ is $10an' + 2(n+1)\Sigma_{i=1}^m |C_i| = k$. It remains to show that $D$ is in fact a $[1,2]$-total dominating set. Every vertex in $\mathcal{V} \cup \mathcal{H}$ is dominated at most twice; once by a vertex in its corresponding forcing gadget and possibly once by a neighbor in $\mathcal{V} \cup \mathcal{H}$. Each vertex in the guard sets is dominated exactly once and each vertex in $\mathcal{W}$, $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ is dominated at most twice (as they have degree two). Finally, each vertex $c_{i,j}$ is dominated exactly once (by a vertex in $\mathcal{W}$), as needed. This concludes the proof of the lemma. $\square$

**Lemma 6.3.** *If $G$ has a $[1,2]$-total dominating set of size at most $k = 10an' + 2(n+1)\Sigma_{i=1}^m |C_i|$ then $\Phi$ is satisfiable.*

**Proof.** Consider a $[1,2]$-total dominating set $D$ of size at most $k$. The set $D$ must contain at least 2 vertex from each forcing gadget. This constitute at least $8an'$ vertices from forcing gadgets. Since $2(n+1)|C_i|$ vertices are required to total dominate the vertices in the clause gadget for $C_i$, we know that $|D \cap (\mathcal{W} \cup \mathcal{X} \cup \mathcal{Y} \cup \mathcal{Z})| \geq 2(n+1)\Sigma_{i=1}^{m}|C_i|$ and, therefore, $|D \cap (\mathcal{V} \cup \mathcal{H} \cup \mathcal{G})| \leq 2an'$. Moreover, $G$ contains $n'$ paths each of length $4a$ and the guard vertices in $\mathcal{G}$ ensure that at least $2a$ vertices are required to dominate the vertices of each path. Finally, since we have $2an'$ pairs of guards with disjoint neighbors, exactly one vertex from $\{v^1_{t,q}, v^2_{t,q}\}$ and one vertex from $\{h^1_{t,q}, h^2_{t,q}\}$ must be in $D$, for each $1 \leq t \leq n'$ and $0 \leq q < a$. This implies that each forcing gadget will contribute two vertices to $D$– the root vertex and one of its neighbor in the 4-cycle. Moreover $|D \cap (\mathcal{W} \cup \mathcal{X} \cup \mathcal{Y} \cup \mathcal{Z})| = 2(n+1)\Sigma_{i=1}^{m}|C_i|$. This implies that exactly two vertex from a path from $c_{i,j}$ to a vertex in $\mathcal{V} \cup \mathcal{H}$ is included in $D$. Moreover, this in turn implies that $c_{i,j} \notin D$ for any $i, j$.

Now, for each $0 \leq q < a$, we construct an assignment $\phi_q$ as follows. For each block $F_t = \{x^1_t, x^2_t\}$, we define $\phi_q(x^1_t)$ as true if $v^1_{t,q} \in D$ and we define $\phi_q(x^1_t)$ to be false if $v^2_{t,q} \in D$. Similarly, for $x^2_t$, we define $\phi_q(x^2_t)$ to be true if $h^1_{t,q} \in D$ and we define $\phi_q(x^2_t)$ to be false if $h^2_{t,q} \in D$. Note that for each block $F_t = \{x^1_t, x^2_t\}$ and each $0 \leq q < a$, we have:

- If $\phi_q(x^1_t)$ is true then $\phi_{q+1}(x^1_t)$ is true. Otherwise, both $v^2_{t,q}$ and $v^1_{t,q+1}$ are not in the dominating set and vertices in $p^2_{t,q}$ is not dominated by $D$.
- If $\phi_q(x^2_t)$ is false then $\phi_{q+1}(x^2_t)$ is false. Otherwise, both $h^2_{t,q}$ and $h^1_{t,q+1}$ are in the dominating set and vertex $v^1_{t,q+1}$ is dominated three times in $D$; by $h^2_{t,q}$, $h^1_{t,q+1}$, and the root of the forcing gadget corresponding to $v^1_{t,q+1}$.

For each variable $x$, we define a sequence $\hat{\phi}_x = \phi_0(x)\phi_1(x)\ldots\phi_{a-1}(x)$. By the discussion above we infer that for each variable $x$, the sequence $\hat{\phi}_x$ can change its value at most once. Hence, as $a = m(n+1)$, we conclude that there exists $0 \leq j < n+1$ such that for all $0 \leq i < m$, the assignment $\phi_{mj+i}(x)$ are equal. We claim that the assignment $\phi = \phi_{mj}(x)$ satisfies $\phi$. Consider a clause $C_i$ and focus on the vertex $c_{i,j}$. We know that $c_{i,j} \notin D$. Thus one of its neighbors from $\mathcal{W}$ (say $w^{\ell'}_{i,j}$) is contained in $D$. Therefore, by the definition of a $[1,2]$-total dominating set, $x^{\ell'}_{i,j}$ is in $D$. We have already argued that any path from $c_{i,j}$ to any vertex in $\mathcal{V} \cup \mathcal{H}$ can contain at most 2 vertices in $D$. This implies that $z^{\ell'}_{i,j}$ is dominated by a vertex in $\mathcal{V} \cup \mathcal{H}$ and the literal corresponding to it will be set to true by $\phi$. This completes the proof of the lemma. □

**Lemma 6.4.** $pw(G) \leq n' + O(1)$.

**Proof.** Let us first recall the definition of a mixed search game. In a mixed search game, the graph $G$ represents a "system of tunnels". Initially, all edges are contaminated by a gas. An edge is cleared by placing searchers at both its endpoints simultaneously or by sliding a searcher along the edge. A cleared edge is re-contaminated if there is a path from an uncleared edge to the cleared edge without any searchers on its vertices or edges. A search is a sequence of operations that can be of the following types: (i) placement of a new searcher on a vertex; (ii) removal of a searcher from a vertex; (iii) sliding a searcher on a vertex along an incident edge and placing the searcher on the other end. A search strategy is winning if after its termination all edges are cleared. The mixed search number of a graph $G$, denoted by $ms(G)$, is the minimum number of searchers required for a winning strategy of mixed searching on G. Takahashi et al. [29] obtained the following relationship between $pw(G)$ and $ms(G)$: $pw(G) \leq ms(G) \leq pw(G) + 1$.

We give a mixed strategy to clean the graph with $n' + O(1)$ searchers. The cleaning process will be done in $a$ rounds. At each round $0 \leq q < a$, we put a searcher on clause variable $c_{i,j}$ and keep this searcher there until the cleaning round is completed. It is clear we could clean each forcing gadget by four searcher and keep searcher on the root vertex after cleaning the edges in the force gadget. For each $1 \leq t \leq n'$ and $0 \leq q < a$, after cleaning four connected forcing gadgets we slide the searchers from the root of forcing gadgets to the vertices $v^1_{t,q}$, $h^1_{t,q}$, $v^2_{t,q}$, $h^2_{t,q}$. Then we put searchers on the (at most two) $z$ vertices connected to them and the guard vertices in sets $p^1_{t,q}$, $p^2_{t,q}$ and $p^3_{t,q}$. As we have a searcher on $c_{i,j}$ until the end of this round we could clean the paths between vertex $c_{i,j}$ and those $z$ vertices using one more searcher. The last step of the round is removing the searchers from vertices $v^1_{t,q}, h^1_{t,q}, v^2_{t,q}, h^2_{t,q}$ and searchers on the guard sets $p^1_{t,q}, p^2_{t,q}$ and $p^3_{t,q}$. We just keep the searcher on $v^1_{t,q+1}$. Also, we remove searchers on the path between $c_{i,j}$ and $\mathcal{V} \cup \mathcal{H}$ except for the one standing on the $c_{i,j}$. To commence the next round, the searcher in $c_{i,j}$ is deleted and a new searcher is put on $c_{i,j+1}$. After the last round the whole graph $G$ is cleaned. Since at any point in time we need at most 14 searchers (they are on $c_{i,j}$, two $z$ vertices, $v^1_{t,q}$, $h^1_{t,q}$, $v^2_{t,q}$, $h^2_{t,q}$, guard vertices in sets $p^1_{t,q}$, $p^2_{t,q}$ and $p^3_{t,q}$, one searcher to clean the paths between two $z$ vertices and $c_{i,j}$) and we reuse 14 searchers in the cleaning process, $n' + 14$ searchers suffice to clean the graph. □

**Theorem 6.5.** *Assuming SETH, for any $\epsilon > 0$ there is no algorithm running in time $(4-\epsilon)^{\mathrm{pw}}|V(G)|^{O(1)}$ for* $[1,2]$-Total Dominating Set *on a graph $G$ with pathwidth* pw.

**Proof.** Suppose that $[1,2]$-Total Dominating Set can be solved in time $(4-\epsilon)^{\mathrm{pw}}|V(G)|^{O(1)}$. Given an instance of CNF-SAT, we can construct an instance of $[1,2]$-Total Dominating Set using the above construction and solve it with a $(4-\epsilon)^{\mathrm{pw}}|V(G)|^{O(1)}$ time algorithm. The correctness of the algorithm follows from Lemmata 6.2 and 6.3. Lemma 6.4 implies

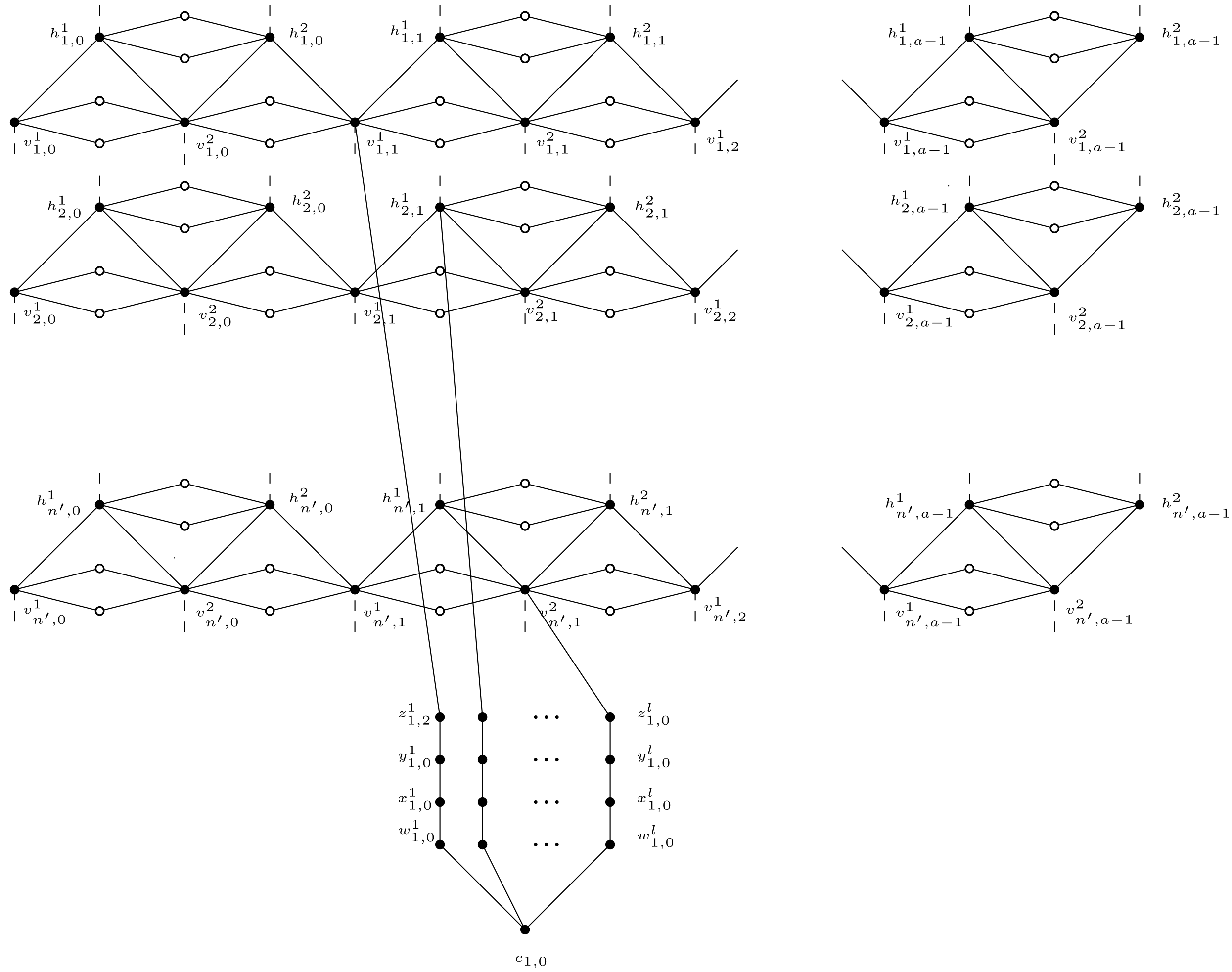


**Fig. 5.** Parts of the construction. Dashed edges are connecting vertices with a forcing gadget.

that the running time of the algorithm is $(4-\epsilon)^{\frac{n}{2}}|V(G)|^{O(1)}$. However, we have $(4-\epsilon)^{\frac{n}{2}} = (\sqrt{4-\epsilon})^n$ and $\sqrt{4-\epsilon} < 2$ which refutes SETH. This concludes the proof. $\square$

## 7. Restrained domination

Recall that a set $D \subseteq V$ of a graph $G$ is called a restrained dominating set if every vertex not in $D$ is adjacent to a vertex in $D$ and a vertex in $V \setminus D$. In this section, we prove the following theorem.

**Theorem 7.1.** *The* Restrained Dominating Set *problem parameterized by the solution size k is W[1]-hard even when restricted to* 3*-degenerate graphs.*

Toward proving the above theorem, we give a reduction from the Multicolored Independent Set problem.

**The reduction.** As in Section 4, we reduce the Multicolored Independent Set problem (with parameter $k$) to the problem of finding a restrained dominating set of size at most $3k+2$ in a graph $G'$ of degeneracy at most 3. Let $k$ be an integer and $G$ be a $k$-colored graph such that its vertices are partitioned into $k$ groups $V_1, V_2, \ldots, V_k$, where each group corresponds to an independent set of the same color. We replace every edge $e = \{u, v\} \in E(G)$ by a length 2 path $uv_ev$, where $v_e$ is a new vertex corresponding to the edge $e$. The set $S_E$ is the collection of all vertices $v_e$ that are added to the graph $G'$. For each group $V_i$, we add a new guard gadget constructed as follows. We add a claw graph (or a star with 4 vertices) centered at $u_i$. We then add a vertex $u'_i$ and add edges between $u'_i$ and the three pendant vertices of the claw. Finally, we add a new vertex of degree one connected to $u'_i$. For each group $V_i$, we add two vertices $x^1_i$ and $x^2_i$. We connect $x^1_i$, $x^2_i$, and $u_i$ to all vertices in $V_i$. We add another claw centered at $r$, connect its pendant vertices to a new vertex $r'$, and

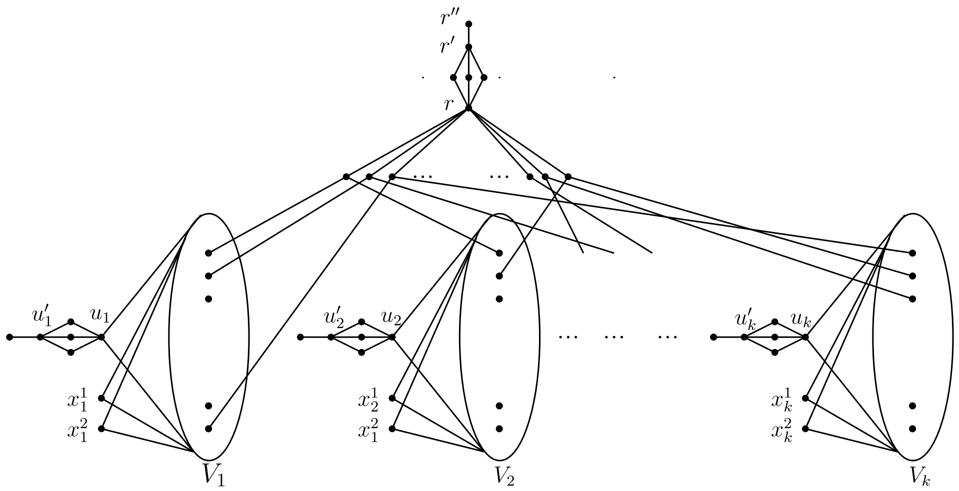


**Fig. 6.** Constructed graph $G$ with degeneracy at most 3.

then connect $r'$ to a new pendant vertex $r''$. Finally, we add edges between $r$ and all of the vertices in $S_E$. This concludes the construction of $G'$. Following simple observation is need to prove the correctness of the reduction (see Fig. 6).

**Observation 7.2.** *Let $D$ be a restrained dominating set of a graph $G$. Then, every vertex $v$ of degree one in $G$ must be in $D$.*

**Lemma 7.3.** *The degeneracy of the constructed graph $G'$ is at most 3.*

**Proof.** We prove the lemma by constructing a degeneracy ordering. First we add all the vertices $S_E$, then we add all the vertices in $V_1 \cup \ldots \cup V_k$ to the order. Then we add the rest of the vertices. It is not hard to see that every vertex has at most 3 neighbors appearing later in the ordering, as needed. $\square$

**Lemma 7.4.** *If there exists a $k$-colored independent set in $G$ then there exists a restrained dominating set of size $3k + 2$ in $G'$.*

**Proof.** Without loss of generality we assume that $|V_i| \geq 2$ for all $i$. Let $S$ be a $k$-colored independent set. We construct a restrained dominating set $D$ as follows. First, we add to $D$ the vertices in $S$. Then we add the pendant vertices (there are $k + 1$ such vertices) to $D$. Then we add $r$ and all $u_i$ vertices to $D$. Clearly $|D| = 3k + 2$. All $x_i^1, x_i^2$ vertices, $1 \leq i \leq k$, are dominated by vertices in the $k$-colored independent set. Since $|V_i \cap D| = 1$, there is also a vertex from the neighborhood of $x_i^1, x_i^2$ which is not in $D$. Every vertex in the guard vertex set (which is not in D) other than $u'_i$ is dominated by $u_i$ and $u'_i$ is one of its neighbor. Similarly, $u'_i$ is dominated by the pendent vertex in the gadget and three of its neighbors are not in $D$. Since $S$ is an independent set, for any vertex in $S_E$ at least one its neighbor is not in $D$, while $r$ is in $D$. This completes the proof. $\square$

**Lemma 7.5.** *If there exists a restrained dominating set of size $3k + 2$ in $G'$ then there exists a $k$-colored independent set in $G$.*

**Proof.** Let $D$ be a restrained dominating set of size $3k + 2$. By Observation 7.2, all the $k + 1$ vertices of degree one are in $D$. Note that at least one additional vertex is required to dominate the remaining vertices in each $k + 1$ guard gadgets. Since $2k + 2$ vertices are already fixed in $D$ and $x_i^1$ and $x_i^2$ are dominated by $D$, we conclude that $|D \cap V_i| = 1$, for each $V_i$. Hence, we can assume, without loss of generality, that vertex $r$ and all the $u_i$ vertices, $1 \leq i \leq k$, are contained in $D$. We claim that the $k$ vertices from $D \cap V(G)$ must be independent. Let $v_1, v_2, \ldots, v_k$ be the vertices in $D \cap V_1$, $D \cap V_2$, …, $D \cap V_k$, respectively. If there exists an edge $e$ between two of those vertices, say $v_l$ and $v_p$, then the vertex $v_e \in S_E$ corresponding to this edge does not have any neighbors in $V \setminus D$. Therefore, the vertices $v_1, v_2, \ldots v_k$ must be independent, as needed. $\square$

## 8. Conclusion

We have shown that the $[1, j]$-Dominating Set problem parameterized by the solution size is W[1]-hard on graphs of degeneracy $(j + 1)$. It is thus natural to ask whether the problem becomes fixed-parameter tractable when the degeneracy of the graph is smaller or equal to $j$. In particular, is the $[1, j]$-Dominating Set problem fixed-parameter tractable on $j$-degenerate graphs? There is a very rich literature [10,11,30] on kernelization for the Dominating Set problem on sparse graphs and it would be interesting to see where (if anywhere) the known techniques for the Dominating Set problem become applicable to the $[1, j]$-Dominating Set problem. Finally, we have shown a lower bound of $O^*((4 - \epsilon)^{\mathrm{tw}})$ for

$[1,2]$-Total Dominating Set assuming SETH, while the known upper bound is $O^*(6^{\mathrm{tw}})$ (which follows from Proposition 6.1). Closing this gap is an interesting open problem. On the other hand $[1,2]$-Dominating Set can be solved in time $O^*(4^{\mathrm{tw}})$ and getting a matching lower bound is another open problem. Moreover, can we get lower bounds for more general problems such as $[1, j]$-Dominating Set and $[1, j]$-Total Dominating Set?

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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