On the proof of the genus bound for Enriques–Fano threefolds

Andreas Leopold Knutsen1,∗, Angelo Felice Lopez2,∗∗ and Roberto Muñoz3,∗∗∗

1Andreas Leopold Knutsen, Department of Mathematics, University of Bergen, Johannes Bruns gate 12, 5008 Bergen, Norway
e-mail: Andreas.Knutsen@math.uib.no

2Angelo Felice Lopez, Dipartimento di Matematica, Università di Roma Tre, Largo San Leonardo Murialdo 1, 00146, Roma, Italy
e-mail: lopez@mat.uniroma3.it

3Roberto Muñoz, ESCET, Departamento de Matemática Aplicada, Universidad Rey Juan Carlos, 28933 Móstoles (Madrid), Spain
e-mail: roberto.munoz@urjc.es

Communicated by: Purna Bangere

Received: July 18, 2011

Abstract. Given an Enriques surface \( S \) embedded in \( \mathbb{P}^r \) with a certain linear system, we show that \( S \) is not hyperplane section of any threefold \( X \subset \mathbb{P}^{r+1} \) that is not a cone over \( S \). This special case completes the proof of the genus bound for Enriques–Fano threefolds [11, Thm. 1.5].


1. Introduction

Given a smooth variety \( Y \subset \mathbb{P}^r \), a very natural question is whether \( Y \) can be hyperplane section of a variety \( X \subset \mathbb{P}^{r+1} \) that is not a cone over \( Y \). When this does not happen \( Y \subset \mathbb{P}^r \) is said to be nonextendable. While several classical works have addressed this question for special classes of

∗Research partially supported by a Marie Curie Intra-European Fellowship within the 6th European Community Framework Programme.

∗∗Research partially supported by the MIUR national project “Geometria delle varietà algebriche” COFIN 2002–2004.

∗∗∗Research partially supported by the MCYT project BFM2003-03971.
varieties \( Y \), in 1989 Zak [20], [15, Thm. 0.1] proved that if \( \text{codim} \ Y \geq 2 \) and \( h^0(N_Y/P_r(-1)) \leq r + 1 \), then \( Y \) is nonextendable. The shift was then on how to compute the cohomology \( h^0(N_Y/P_r(-1)) \). In the same year a result of Wahl [19, Prop. 1.10] introduced, for this goal, when \( Y \) is a curve, the point of view of Gaussian maps: 

\[
h^0(N_Y/P_r(-1)) = r + 1 + \text{cork} \Phi_{H_{\alpha_{\text{can}}}} \]

where \( \Phi_{H_{\alpha_{\text{can}}}} \) is the Gaussian map associated to the canonical and hyperplane bundle \( H_Y \) of \( Y \). The pairing of these two results led to a number of articles about Gaussian maps and nonextendability of curves. In some notable cases this could also be extended to study nonextendability of surfaces, by passing to their curve section. We mention, for example, [2,3] where \( Y \) was a general hyperplane section of a general K3 surface.

On the other hand, until the introduction of [11, Thm. 1.1], no general method was known when \( Y \) is a surface. This method, still based on Gaussian maps on suitable linear systems on \( Y \), was applied in [11] to study nonextendability of pluricanonical embeddings of surfaces of general type and of Enriques surfaces. The study of the latter case led then in [11] to give a genus bound (also obtained simultaneously and independently by Prokhorov [17]) for the curve section, in analogy with the case of smooth Fano threefolds ([9,10,16,2,3]), for Enriques–Fano threefolds, that is threefolds \( X \subset \mathbb{P}^N \) having a hyperplane section \( Y \) that is a smooth Enriques surface, and such that \( X \) is not a cone over \( Y \). Such threefolds were classically studied by Fano [8] and more recently by Conte and Murre [4] and specially by Bayle [1, Thm. A] and Sano [18, Thm. 1.1], but all of these works were not enough to get a genus bound.

The genus bound in [11, Thm. 1.5] depends therefore on studying nonextendability of Enriques surfaces embedded in \( \mathbb{P}^r \) with a very ample line bundle. Now in the course of the proof of [11, Thm. 1.5], one explicit case of embedding line bundle (see [11, Prop. 8.5]) was not treated, for reasons of space. In the present article we therefore complete the proof by showing nonextendability in this case.

To be more precise let now \( S \subset \mathbb{P}^r \) be an Enriques surface embedded by a very ample line bundle \( H \) of degree \( H^2 = 2g - 2 \). Let \( E > 0 \) be such that \( E^2 = 0 \) and \( E \cdot H = \phi(H) \) (see Definition 2.3).

Suppose that \( H \) be of type (I), that is, after applying the decomposition procedure of [11, §6] (briefly recalled at the beginning of Section 3), we get an effective decomposition

\[ H \equiv \beta E + \gamma E_1 + M_2 \]

with \( E^2 = E_1^2 = 0 \) and \( E \cdot E_1 = 1 \).

Then we have

**Theorem 1.1.** Let \( H \) be of type (I) with \( \beta \leq 4 \), \( \gamma = 2 \) and \( M_2 > 0 \) and such that \( H^2 \geq 32 \) or \( H^2 = 28 \). Then \( S \) is nonextendable, except possibly for the following two cases, where \( H^2 = 28 \) and \( E_2 := M_2, E_2^2 = 0 \):
(i) $H \sim 3E + 2E_1 + E_2$, $E \cdot E_1 = E_1 \cdot E_2 = 1$, $E \cdot E_2 = 2$.

(ii) $H \sim 4E + 2E_1 + E_2$, $E \cdot E_1 = E \cdot E_2 = E_1 \cdot E_2 = 1$.

The above theorem proves [11, Prop. 8.5] and therefore completes the proof of the genus bound for Enriques–Fano threefolds [11, Thm. 1.5]. As a suggestion to the reader, we remark that the present article and [11], at least in the part regarding nonextendability of Enriques surfaces, are written to be read together.

2. Basic facts on line bundles on Enriques surfaces

We first recall.

**Definition 2.1.** Let $L$ and $M$ be line bundles on a smooth projective variety. Given $V \subseteq H^0(L)$ we denote by $\mu_{V,M} : V \otimes H^0(M) \to H^0(L \otimes M)$ the multiplication map of sections, $\mu_{L,M}$ when $V = H^0(L)$, and by $\Phi_{L,M} : \text{Ker} \mu_{L,M} \to H^0(\Omega^1_X \otimes L \otimes M)$ the Gaussian map. This map can be defined locally by $\Phi_{L,M}(s \otimes t) = sdt - tds$ [19, 1.1].

We henceforth let $S$ be an Enriques surface.

**Definition 2.2.** We denote by $\sim$ (respectively $\equiv$) the linear (respectively numerical) equivalence of divisors (or line bundles) on $S$. A line bundle $L$ on $S$ is **primitive** if $L \equiv hL'$ for some line bundle $L'$ and some integer $h$, implies $h = \pm 1$. An effective line bundle $L$ on $S$ is **quasi-nef** [12] if $L^2 \geq 0$ and $L \cdot \Delta \geq -1$ for every $\Delta$ such that $\Delta > 0$ and $\Delta^2 = -2$.

A **nodal curve** on $S$ is a smooth rational curve. A **nodal cycle** on $S$ is a divisor $R > 0$ such that $(R')^2 \leq -2$ for any $0 < R' \leq R$. An **isotropic divisor** $F$ on $S$ is a divisor such that $F^2 = 0$ and $F \not\equiv 0$. An **isotropic $k$-sequence** is a set $\{f_1, \ldots, f_k\}$ of isotropic divisors such that $f_i \cdot f_j = 1$ for $i \neq j$.

We will often use the fact that if $R$ is a nodal cycle, then $h^0(R) = 1$ and $h^0(R + K_S) = 0$.

**Definition 2.3.** Let $L$ be a line bundle on $S$ with $L^2 > 0$. Following [6] we define

$$\phi(L) = \inf\{|F \cdot L| : F \in \text{Pic} S, F^2 = 0, F \not\equiv 0\}.$$ 

One has $\phi(L)^2 \leq L^2$ [6, Cor. 2.7.1] and, if $L$ is nef, then there exists a genus one pencil $|2E|$ such that $E \cdot L = \phi(L)$ [5, 2.11]. Moreover we will often use the fact that if $L$ is nef, then it is base-point free if and only if $\phi(L) \geq 2$ [6, Prop. 3.1.6, 3.1.4 and Thm. 4.4.1].
A line bundle $L > 0$ with $L^2 \geq 0$ on $S$ has a (nonunique) decomposition $L \equiv a_1 E_1 + \cdots + a_n E_n$, where $a_i$ are positive integers, and each $E_i$ is primitive, effective and isotropic, cf. e.g. [13, Lemma 2.12]. We will call such a decomposition an arithmetic genus 1 decomposition. An effective line bundle $L$ on $S$ with $L^2 \geq 0$ is said to be of small type if either $L = 0$ or if in every arithmetic genus 1 decomposition of $L$ as above, all $a_i = 1$.

Line bundles of small type have specific decompositions that are classified in [11, Lemma 4.3].

We also record the following two useful results.

**Lemma 2.5.** Let $L$ be a nef and big line bundle on an Enriques surface and let $F$ be a divisor satisfying $F \cdot L < 2\phi(L)$ (respectively $F \cdot L = \phi(L)$ and $L$ is ample). Then $h^0(F) \leq 1$ and if $F > 0$ and $F^2 \geq 0$ we have $F^2 = 0$, $h^0(F) = 1$, $h^1(F) = 0$ and $F$ is primitive and quasi-nef (resp. nef).

**Proof.** If $h^0(F) \geq 2$ we can write $|F| = |M| + G$, with $M$ the moving part and $G \geq 0$ the fixed part of $|F|$. By [6, Prop. 3.1.4] we get $F \cdot L \geq 2\phi(L)$, a contradiction. Then $h^0(F) \leq 1$ and if $F > 0$ and $F^2 \geq 0$ it follows that $F^2 = 0$ and $h^1(F) = 0$ by Riemann-Roch. Hence $F$ is quasi-nef and primitive by [12, Cor. 2.5]. If $F \cdot L = \phi(L)$, $L$ is ample and $F$ is not nef, by [13, Lemma 2.4] we can write $F \sim F_0 + \Gamma$ with $F_0 > 0$, $F_0^2 = 0$ and $\Gamma$ a nodal curve. But then $F_0 \cdot L < \phi(L)$.

**Lemma 2.6.** For $1 \leq i \leq 4$ let $F_i > 0$ be four isotropic divisors on $S$ such that $F_1 \cdot F_2 = F_3 \cdot F_4 = 1$ and $F_1 \cdot F_3 = F_2 \cdot F_3 = 2$. If $F_4 \cdot (F_1 + F_2) = 4$ then $F_1 \cdot F_4 = F_2 \cdot F_4 = 2$.

**Proof.** By symmetry and [12, Lemma 2.1] we can assume, to get a contradiction, that $F_1 \cdot F_4 = 1$ and $F_2 \cdot F_4 = 3$. Then $(F_2 + F_4)^2 = 6$ and $\phi(F_2 + F_4) = 2$ whence, by [13, Lemma 2.4], we can write $F_2 + F_4 \sim A_1 + A_2 + A_3$ with $A_i > 0$, $A_i^2 = 0$ and $A_i \cdot A_j = 1$ for $i \neq j$. But this gives the contradiction $8 = (F_2 + F_4) \cdot (F_1 + F_2 + F_3) \geq 3\phi(F_1 + F_2 + F_3) = 9$.

### 3. First reductions in the proof of Theorem 1.1

In this section we show how to use some results in [11] to reduce the proof of Theorem 1.1 to some explicit intersections cases (Lemma 3.1).

We briefly recall here the decomposition procedure of [11, §6].

Let $S \subset \mathbb{P}^n$ be an Enriques surface of sectional genus $g$ and let $H$ be its hyperplane divisor. Let $|2E|$ be a genus one pencil such that $E \cdot H = \phi(H)$ and, as $H$ is not of small type by [11, Lemma 4.3], we can define, as in [11, §4],

\[ E_i \]
\[
\alpha = \min\{k \geq 2| (H - kE)^2 \geq 0 \text{ and if } (H - kE)^2 > 0 \text{ there exists } F > 0 \text{ with } F^2 = 0, F \cdot E > 0 \text{ and } F \cdot (H - kE) \leq \phi(H)\},
\]

\[L_1 = H - \alpha E \text{ and let } E_1 > 0 \text{ be such that } E_1^2 = 0 \text{ and } E_1 \cdot L_1 = \phi(L_1).\]

Now repeat the procedure on \(L_1\). Then we get a decomposition

\[H = \alpha E + \alpha_1 E_1 + \alpha_2 E_2' + \cdots + \alpha_{n-1} E_{n-1}' + L_n,
\]

for some \(n \geq 1, \alpha, \alpha_i \geq 2\) for \(1 \leq i \leq n - 1\) and \(L_n\) is of small type.

Removing copies of \(E\) or \(E_1\) from \(L_n\) one gets several decompositions (see [11, §6]).

We say that the decomposition is of type (I) if \(H\) is not 2-divisible in \(\text{Num}(S)\) and we are in one of the two cases

(I-A) \(n = 3, E_2' \equiv E\), or

(I-B) \(n = 2\).

This allows us to write

\[H \equiv \beta E + \gamma E_1 + M_2, \text{ with } E \cdot E_1 = 1.
\]

Note that, in particular, when \(\beta \leq 3\), we must be in case (I-B).

We now start the proof of Theorem 1.1.

Let \(H\) be as in Theorem 1.1. Replacing \(M_2\) with \(M_2 + K_S\), that has the same properties, we can assume

\[H \sim \beta E + 2E_1 + M_2.
\]

Since by construction \(M_2\) neither contains \(E\) nor \(E_1\) in its arithmetic genus 1 decompositions, we have \((M_2 - E)^2 < 0\) and \((M_2 - E_1)^2 < 0\). Also \(E \cdot H \leq E_1 \cdot H\) and \(E_1 \cdot L_1 \leq E \cdot L_1\), giving

\[
\frac{1}{2} M_2^2 + 1 \leq E \cdot M_2 \leq E_1 \cdot M_2 + \beta - 2, \quad \text{and} \quad (1)
\]

\[
\frac{1}{2} M_2^2 + 1 \leq E_1 \cdot M_2 \leq E \cdot M_2 + 2 - \beta + a \leq E \cdot M_2 + 2. \quad (2)
\]

Also, by [11, Lemmas 6.1 and 6.2], we have

\[E + E_1 \text{ is base-component free. If } \Delta > 0 \text{ is such that } \Delta^2 = -2 \text{ and } \Delta \cdot E_1 < 0, \text{ then } \Delta \text{ is a nodal curve and } E_1 \sim E + \Delta + K_S. \quad (3)
\]

Now we can give a first reduction.
Lemma 3.1. Let $H$ be of type (I) with $\beta \leq 4$, $\gamma = 2$ and $M_2^2 \geq 2$. Then $S$ is nonextendable unless, possibly, we are in one of the following cases (where all the $E_i$'s are effective and isotropic):

(a) $M_2^2 = 2$, $M_2 \sim E_2 + E_3$, $E_2 \cdot E_3 = 1$, and either

(\text{a-i}) $\beta = 2$, $(E \cdot E_2, E \cdot E_3, E_1 \cdot E_2, E_1 \cdot E_3) = (1, 2, 1, 2), (1, 2, 2, 1)$, (1, 2, 2, 2); or
(\text{a-ii}) $\beta = 3$, $(E \cdot E_2, E \cdot E_3, E_1 \cdot E_2, E_1 \cdot E_3) = (2, 2, 2, 2), (2, 2, 1, 2);$
(\text{a-iii}) $\beta = 3$, 4, $(E \cdot E_2, E \cdot E_3, E_1 \cdot E_2, E_1 \cdot E_3) = (1, 1, 1, 1), (1, 1, 2), (1, 1, 2, 2)$. 

(b) $M_2^2 = 4$, $M_2 \sim E_2 + E_3$, $E_2 \cdot E_3 = 2$, and either

(\text{b-i}) $\beta = 2$, $(E \cdot E_2, E \cdot E_3, E_1 \cdot E_2, E_1 \cdot E_3) = (1, 2, 1, 2), (1, 2, 2, 1)$, (1, 2, 2, 3); or
(\text{b-ii}) $\beta = 3$, $(E \cdot E_2, E \cdot E_3, E_1 \cdot E_2, E_1 \cdot E_3) = (1, 2, 2, 1), (1, 2, 1, 3)$. 

(c) $M_2^2 = 6$, $M_2 \sim E_2 + E_3 + E_4$, $E_2 \cdot E_3 = E_2 \cdot E_4 = E_3 \cdot E_4 = 1$, and

$\beta = 2$, $(E \cdot E_2, E \cdot E_3, E \cdot E_4, E_1 \cdot E_2, E_1 \cdot E_3, E_1 \cdot E_4) = (1, 1, 2, 1, 2)$. 

Proof. We write $M_2 \sim E_2 + \cdots + E_{k+1}$ as in [11, Lemma 4.3] with $k = 2$ or 3. Moreover we can assume that $1 \leq E \cdot E_2 \leq \cdots \leq E \cdot E_{k+1}$, whence that $E \cdot M_2 \geq kE \cdot E_2$.

We first consider the case $\beta = 4$.

We note that $(M_2 - 2E_2)^2 = -2$ if $M_2^2 = 2$ or 6, $(M_2 - 2E_2)^2 = -4$ if $M_2^2 = 4$ and $(M_2 - 2E_2)^2 \geq -6$ if $M_2^2 = 10$. In the latter case $E \cdot M_2 \geq 6$ by (1), whence $E \cdot (M_2 - 2E_2) \geq 2$. Using this and setting $B := E + E_1 + E_2$ one easily verifies that $(H - 2B)^2 = 4E \cdot (M_2 - 2E_2) + (M_2 - 2E_2)^2 \geq 0$ and $E \cdot (H - 2B) > 0$ (whence $H - 2B \geq 0$ by Riemann-Roch), except for the cases

$$M_2^2 = 2, 4 \quad \text{and} \quad E \cdot E_2 = E \cdot E_3. \quad (4)$$

Moreover, except for these cases, using (1) and (2), one easily verifies that $H^2 \geq 54$, except for the case $M_2^2 = 2$ and $(E \cdot M_2, E_1 \cdot M_2) = (3, 2)$, where $H^2 = 50$. In this case $(3B - H) \cdot H = 4 < \phi(H) = 5$, so that, if $3B - H > 0$ it must be a nodal cycle. Therefore either $h^0(3B - H) = 0$ or $h^0(3B + K_S - H) = 0$, so in any case $B$ satisfies the conditions in [11, Prop. 5.2] or in [11, Prop. 5.3] and $S$ is nonextendable.

In the remaining cases (4) we can without loss of generality assume $1 \leq E_1 \cdot E_2 \leq E_1 \cdot E_3$ and we set $B := E + E_2$. Then $(H - 2B)^2 = 8 + 4E_1 \cdot (E_3 - E_2) + (E_3 - E_2)^2 \geq 4$ and $(H - 2B) \cdot E = 2$. Using (1) and (2), one gets $H^2 \geq 64$ if $M_2^2 = 4$, $H^2 \geq 74$ if $M_2^2 = 2$ and $E \cdot E_2 = E \cdot E_3 = 3$, and
$B \cdot H \geq 17$ if $M^2_2 = 2$ and $E \cdot E_2 = E \cdot E_3 = 2$. Moreover, in the latter case, we have that again $H^2 \geq 64$ unless $E_1 \cdot M_2 = 2, 3$, which gives $E_1 \cdot E_2 = 1$ and $B$ is nef by [11, Lemma 6.3(c)] since $E_2 \cdot H = 11 < 2\phi(H) = 12$, whence $E_2$ is quasi-nef by Lemma 2.5. Therefore $B$ satisfies the conditions in [11, Prop. 5.2] or in [11, Prop. 5.4] and $S$ is nonextendable unless $M^2_2 = 2$ and $E \cdot E_2 = E \cdot E_3 = 1$. In the latter case, by (2) we have $2 \leq E_1 \cdot M_2 \leq 4$ and $E_1 \cdot M_2 = 4$ then $\alpha = 4$. In this last case $L_1 \sim 2E_1 + M_2$, whence $\phi(L_1) = E_1 \cdot M_2 = 4$ and we get that $4 \leq E_i \cdot L_1 = 2E_1 \cdot E_i + 1$ for $i = 2, 3$, so that $E_1 \cdot E_2 = E_1 \cdot E_3 = 2$. Therefore we get the cases in (a-iii) with $\beta = 4$.

We next treat the cases $\beta \leq 3$. As we know, we are in case (1-B), whence $L_2$ is of small type and either $L_2 \sim M_2$ or $L_2 \sim E + M_2$.

Suppose first that $L_2 \sim E + M_2$.

Then $\beta = 3$, $\alpha = 2$ and, since $L_2$ is of small type, by (1), we can only have $(M^2_2, E \cdot M_2) = (2, 2), (2, 4)$ or $(4, 3)$.

If $(M^2_2, E \cdot M_2) = (2, 2)$, then $E \cdot E_2 = E \cdot E_3 = 1$ and by (2) we have $2 \leq E_1 \cdot M_2 \leq 3$, yielding the first two cases in (a-iii).

If $(M^2_2, E \cdot M_2) = (2, 4)$, then $L^2_2 = 10$ and $\phi(L_2) = 3$. As $E \cdot E_i + 1 = L_2 \cdot E_i \geq \phi(L_2) = 3$ for $i = 2, 3$, we must have $E \cdot E_2 = E \cdot E_3 = 2$. Now $L_1 \sim E + 2E_1 + M_2$ and $(1 + E_1 \cdot M_2)^2 = \phi(L_1)^2 \leq L^2_1 = 14 + 4E_1 \cdot M_2$ and (1) yield $E_1 \cdot M_2 = 3$ or 4. Therefore, by Lemma 2.6 and symmetry, we get the two cases in (a-ii).

If $(M^2_2, E \cdot M_2) = (4, 3)$, then $E \cdot E_2 = 3$ or 4 by (2). Since $L^2_2 = 10$ and $\phi(L_1) = E \cdot L_2 = 3$, there is by [6, Cor. 2.5.5] an isotropic effective 10-sequence $(f_1, \ldots, f_{10})$ such that $E = f_1$ and $3L_2 \sim f_1 + \cdots + f_{10}$.

In the case $E_1 \cdot M_2 = 3$ we get $E_1 \cdot L_2 = 4$, whence we can assume, possibly after renumbering, that $E_1 \cdot f_i = 1$ for $1 \leq i \leq 8$ and $(E_1 \cdot f_9, E_1 \cdot f_{10}) = (2, 2)$ or (1, 3). In the latter case we have $(E_1 + f_{10})^2 = 6$ and $\phi(E_1 + f_{10}) = 2$, whence we can write $E_1 + f_{10} \sim A_1 + A_2 + A_3$ for some $A_i > 0$ such that $A_i^2 = 0, A_i \cdot A_j = 1$ for $i \neq j$. But $f_i \cdot (E_1 + f_{10}) = 2$ for all $1 \leq i \leq 9$, a contradiction. Hence $(E_1 \cdot f_9, E_1 \cdot f_{10}) = (2, 2)$. One easily sees that there is an isotropic divisor $f_{19} > 0$ such that $f_{19} \cdot f_i = f_{19} \cdot f_9 = 2$ and $L_2 \sim f_1 + f_9 + f_{19}$. Therefore $E_1 \cdot f_{19} = 1$. Setting $E'_2 = f_9$ and $E'_3 = f_{19}$ we get the first case in (b-ii).

If $E_1 \cdot M_2 = 4$ we get $E_1 \cdot L_2 = 5$, whence we can assume, possibly after renumbering, that $E_1 \cdot f_i = 1$ for $1 \leq i \leq 5$. As above there is an isotropic divisor $f_{12} > 0$ such that $f_{12} \cdot f_i = f_{12} \cdot f_2 = 2$ and $L_2 \sim f_1 + f_2 + f_{12}$. Hence $E_1 \cdot f_{12} = 3$. Setting $E'_2 = f_2$ and $E'_3 = f_{12}$ we get the second case in (b-ii).

Finally, we have left the case with $L_2 \sim M_2$, where $\beta = \alpha$. We have $L_1 \sim 2E_1 + M_2$, whence $(E_1 \cdot M_2)^2 = \phi(L_1)^2 \leq L^2_1 = 4E_1 \cdot M_2 + M^2_2$, so that (2) and [13, Prop. 1] give $E_1 \cdot M_2 \leq 4$. In particular $M^2_2 \leq 6$ by (2).
If $\beta = \alpha = 3$, by definition of $\alpha$, we have $1 + E_1 \cdot M_2 = E_1 \cdot (L_1 + E) > \phi(H) = 2 + E \cdot M_2$, whence $E_1 \cdot M_2 = 4$, $E \cdot M_2 = 2$ and $M_2^2 = 2$ by (1). Then $E \cdot E_2 = E \cdot E_3 = 1$ and, for $i = 2, 3, E_i \cdot L_1 = 2E_i, E_1 + 1 \geq \phi(L_1) = E_1 \cdot M_2 = 4$, whence $E_1 \cdot E_2 = E_1 \cdot E_3 = 2$, that is the third case in (a-iii).

In the remaining cases we have $\beta = \alpha = 2$.

If $M_2^2 = 2$ using again $\phi(L_1)^2 \leq L_1^2$, $E_1 \cdot L_1 \geq \phi(L_1)$, (1) and (2) together with $H^2 \geq 32$ or $H^2 = 28$, we deduce the possibilities $(E \cdot M_2, E_1 \cdot M_2) = (3, 3), (2, 4), (3, 4) or (4, 4)$. By symmetry one easily sees that one gets the cases in (a-i).

If $M_2^2 = 4$ we similarly get $(E \cdot M_2, E_1 \cdot M_2) = (3, 3), (3, 4) or (4, 4)$. From the first two cases, using Lemma 2.6 for the second, we obtain the cases in (b-i). If $(E \cdot M_2, E_1 \cdot M_2) = (4, 4)$, we now show that $H$ also has a decomposition of type (III) as in [11, §6]. It will follow that $S$ is nonextendable by [11, §10]. We have $E \cdot H = 6$, whence $(H - 3E)^2 = 8$ and $H - 3E > 0$ by [13, Lemma 2.4]. If $\phi(H - 3E) = 1$ we can write $H - 3E = 4A_1 + A_2$ with $A_j > 0$, $A_1^2 = 0$ and $A_1 \cdot A_2 = 1$. Now $6 \leq H \cdot A_1 = 3E \cdot A_1 + 1$ gives $E \cdot A_1 \geq 2$, whence the contradiction $6 = H \cdot E = 4E \cdot A_1 + E \cdot A_2 \geq 8$.

Therefore there is an $E'_1 > 0$ such that $(E'_1)^2 = 0$ and $E'_1 \cdot (H - 3E) = 2$. Since $(H - 3E - 2E'_1)^2 = 0$, by [13, Lemma 2.4] we can write $H \sim 3E + 2E'_1 + E''_1$, with $E'_2 > 0$, $(E''_1)^2 = 0$ and $E'_1 \cdot E'_2 = 2$. From $6 \leq H \cdot E'_1 = 3E \cdot E'_1 + 2$ we get $E \cdot E'_1 \geq 2$. Now from $6 = H \cdot E = 2E \cdot E'_1 + E \cdot E'_2$ we see that we cannot have $E \cdot E'_1 \geq 3$, for then $E \cdot E'_1 = 3$, $E \cdot E'_2 = 0$, but this gives $E'_1 = 0$. So for some $q \geq 1$ by [12, Lemma 2.1], whence the contradiction $2 = E'_1 \cdot E'_2 = 3q$. Therefore $E \cdot E'_1 = 2$, $E_1 \cdot E'_2 = 1$ so that $E'_2$ is primitive and since $E'_1 \cdot L_1 = E'_1 \cdot (H - 3E) + E'_1 \cdot E = 4 = \phi(L_1)$ we obtain a decomposition of $H$ of type (III), as claimed.

If $M_2^2 = 6$, by (1) and (2) we get, as above, $E_1 \cdot M_2 = E \cdot M_2 = 4$, yielding by symmetry the case in (c) in addition to the case $(E \cdot E_2, E \cdot E_3, E \cdot E_4, E_1 \cdot E_2, E_1 \cdot E_3, E_1 \cdot E_4) = (1, 1, 2, 1, 1, 1, 1)$. In the latter case we note that $\phi(H) = E \cdot H = E_1 \cdot H = 6$ and $\phi(H - 2E_1) = \phi(2E + E_2 + E_3 + E_4) = E_3 \cdot (H - 2E_1) = 4$. Hence we can decompose $H$ with respect to $E_1$ and $E_3$, which means that $H$ is also of type (III) (as in [11, §6]) and $S$ is nonextendable by [11, §10].

4. Conclusion of the proof of Theorem 1.1

By Lemma 3.1 we can assume that either $M_2^2 = 0$ or we are in one of the cases of that lemma. Moreover recall that $H$ is not 2-divisible in Num$(S)$ and we are in case (I-A) or (I-B).

4.1 The case $M_2^2 = 0$

We write $M_2 = E_2$ for a primitive $E_2 > 0$ with $E_2^2 = 0$. 
4.1.1 $\beta = 2$

From (1) and (2) we get $1 \leq E \cdot E_2 \leq E_1 \cdot E_2 \leq E \cdot E_2 + 2$. Moreover, since $L_1 \sim 2E_1 + E_2$, we get $(\phi(L_1))^2 = (E_1 \cdot E_2)^2 \leq L_1^2 = 4E_1 \cdot E_2$, whence $E_1 \cdot E_2 \leq 3$ by [13, Prop. 1], as $E_2$ is primitive. Since $H^2 \geq 28$, we are left with the cases $(E \cdot E_2, E_1 \cdot E_2) = (2, 3)$ or $(3, 3)$, so that $S$ is nonextendable by [11, Lemma 5.5(iii-b)].

4.1.2 $\beta = 3$

From (1) and (2) we get $1 \leq E \cdot E_2 \leq E_1 \cdot E_2 + 1 \leq E \cdot E_2 + \alpha$.

If $\alpha = 2$ we get $E \cdot E_2 - 1 \leq E_1 \cdot E_2 \leq E \cdot E_2 + 1$. Moreover, since we are in case (I-B), $L_2 \sim E + E_2$ is of small type, whence $E \cdot E_2 \leq 3$ or $E \cdot E_2 = 5$. Furthermore, since $L_1 \sim E + 2E_1 + E_2$, we get $(\phi(L_1))^2 = (1 + E_1 \cdot E_2)^2 \leq L_1^2 = 4 + 4E_1 \cdot E_2 + 2E \cdot E_2$. However, in the case $(E \cdot E_2, E_1 \cdot E_2) = (3, 4)$, we find $(L_1^2, \phi(L_1)) = (26, 5)$, which is impossible by [13, Prop. 1]. This yields that $E \cdot E_2 = 2, 3, 5$ if $E_1 \cdot E_2 = E \cdot E_2 - 1$; $E \cdot E_2 = 1, 2, 3$ if $E_1 \cdot E_2 = E \cdot E_2$; and $E \cdot E_2 = 1, 2$ if $E_1 \cdot E_2 = E \cdot E_2 + 1$.

If $\alpha = 3$ we must have, by [11, (11)], that $E_1 \cdot (H - 3E) = \phi(H)$, whence $E_1 \cdot E_2 = 2 + E \cdot E_2$. Moreover, since $L_1 \sim 2E_1 + E_2$, we get $(\phi(L_1))^2 = (E_1 \cdot E_2)^2 \leq L_1^2 = 4E_1 \cdot E_2$, whence $E_1 \cdot E_2 \leq 3$ by [13, Prop. 1] since $E_2$ is primitive. Hence $E_1 \cdot E_2 = 3$ and $E \cdot E_2 = 1$.

To summarize, using $H^2 \geq 32$ or $H^2 = 28$, we have the following cases:

\[
E_1 \cdot E_2 = E \cdot E_2 - 1, \quad E \cdot E_2 = 2, 3 \text{ or } 5, \quad g = 15, 20 \text{ or } 30.
\]

\[
E_1 \cdot E_2 = E \cdot E_2, \quad E \cdot E_2 = 2 \text{ or } 3, \quad g = 17 \text{ or } 22.
\]

\[
E_1 \cdot E_2 = 3, \quad E \cdot E_2 = 2, \quad g = 19. \quad (5)
\]

We will now show, in Lemmas 4.1–4.4, that $S$ is nonextendable in the five cases of genus $g \geq 17$. The case with $g = 15$ is case (i) in Theorem 1.1.

**Lemma 4.1.** In the case $(E \cdot E_2, E_1 \cdot E_2, g) = (5, 4, 30)$ in (5), $S$ is nonextendable.

**Proof.** We have $H^2 = 58$ and $\phi(H) = E \cdot H = E_1 \cdot H = 7$. Hence both $E$ and $E_1$ are nef by Lemma 2.5. Let now $H' = H - 4E$. Then $(H')^2 = 2$ and consequently we can write $H \sim 4E + A_1 + A_2$ for $A_1 \cdot H = 1$ and $A_1 \cdot A_2 = 1$. Since $E \cdot H = E \cdot A_1 + E \cdot A_2 = 7$ we can assume by symmetry that either (a) $(E \cdot A_1, E \cdot A_2) = (2, 5)$ or (b) $(E \cdot A_1, E \cdot A_2) = (3, 4)$. Also since $E_1 \cdot H = 7$ we have $E_1 \cdot (A_1 + A_2) = 3$, whence we have the two possibilities $(E_1 \cdot A_1, E_1 \cdot A_2) = (2, 1)$ or $(1, 2)$.

In case (b) we get $A_1 \cdot H = 13$, whence $(H - 2(E + A_1))^2 = 2$. Since $(H - 2(E + A_1)) \cdot E = 1$, we have $H - 2(E + A_1) > 0$ by Riemann-Roch, whence $S$ is nonextendable by [11, Prop. 5.2].
In case (a) we get $A_1 \cdot H = 9$. Now if $E_1 \cdot A_1 = 2$, we get $(H - 2(E + A_1 + E_1))^2 = 6$, and as above $S$ is nonextendable by [11, Prop. 5.2]. If $E_1 \cdot A_1 = 1$, then $E_1 \cdot (H - 2E) = A_1 \cdot (H - 2E) = 5$, whence $L_1 \sim H - 2E$ and $\phi(L_1) = A_1 \cdot L_1 = 5$. Therefore we can continue the decomposition with respect to $A_1$ instead of $E_1$. Since $H$ now is of type (III) (as in [11, §6]), $S$ is nonextendable by [11, §10].

\[ \square \]

Claim 4.2. Let $H \sim 3E + 2E_1 + E_2$ be as in (5) with $(E \cdot E_2, E_1 \cdot E_2, g) = (3, 2, 20)$ (respectively $(E \cdot E_2, E_1 \cdot E_2, g) = (3, 3, 22)$). Then there exists an isotropic effective 5-sequence $\{E, F_1, F_2, F_3, F_4\}$ (respectively an isotropic effective 4-sequence $\{E, F_1, F_2, F_3\}$) together with an isotropic divisor $F_4 > 0$ such that $E \cdot F_4 = F_2 \cdot F_4 = F_3 \cdot F_4 = 1$ and $F_1 \cdot F_4 = 2$ such that $H \sim 2E + 2F_1 + F_2 + F_3 + F_4$ and:

(a) $F_1$ is nef and $F_i$ is quasi-nef for $i = 2, 3, 4$;
(b) $|E + F_2|$ and $|F_1 + F_3|$ are without base components;
(c) $|E + F_1 + F_2 + F_3|$ and $|E + F_1 + F_4|$ are base-point free;
(d) $h^1(F_1 + F_4 - F_2) = h^2(F_1 + F_4 - F_2) = 0$.

Proof. Since $(E + E_2)^2 = 6$ and both $E$ and $E_2$ are primitive, we can write $E + E_2 \sim A_1 + A_2 + A_3$ with $A_1 > 0$, $A_i^2 = 0$ and $A_i \cdot A_j = 1$ for $i \neq j$.

We easily find (possibly after renumbering) that $A_i \cdot E = A_i \cdot E_2 = A_i \cdot E_1 = A_2 \cdot E_1 = 1$ for $i = 1, 2, 3$ and $A_3 \cdot E_1 = 1$ if $g = 20$ and 2 if $g = 22$. Moreover $A_i \cdot H \leq 8 < 2\phi(H) = 10$, whence all the $A_i$’s are quasi-nef by Lemma 2.5.

Assume now there is a nodal curve $R_i$ with $R_i \cdot A_i = -1$ for $(i, g) \neq (3, 22)$. Then we can as usual write $A_i \sim B_i + R_i$, with $B_i > 0$ primitive and isotropic. Since $A_1 \cdot H = 6$ we deduce that $B_i \equiv E$ or $B_i \equiv E_1$, where the latter case only occurs if $g = 20$.

If $g = 20$, then, since for $i \neq j$, we have $(E + R_i) \cdot (E + R_j) = 2 + R_i \cdot R_j = (E_1 + R_i) \cdot (E_1 + R_j)$, we see that at most two of the $A_i$’s can be not nef, otherwise we would get $R_i \cdot R_j = -1$, a contradiction. Possibly after reordering the $A_i$’s and adding $K_S$ to two of them, we can therefore assume that $A_1$ is nef, and that either $A_2$ is nef or $A_2 \sim E + R + K_S$ for $R$ a nodal curve with $E \cdot R = 1$. Now $E_1$ is nef, by Lemma 2.5, as $E_1 \cdot H = \phi(H) = 5$, so that both $|E_1 + A_1|$ and $|E + A_2|$ are without fixed components. Setting $F_1 = E_1$, $F_2 = A_2$, $F_3 = A_1$ and $F_4 = A_3$ we therefore have the desired decomposition satisfying (a) and (b). It also follows by construction that $E + F_1 + F_2 + F_3$ and $E + F_1 + F_4$ are nef, the latter because $E$ and $F_1$ are, and $F_4$ is either nef or $F_4 \equiv A + R'$ with $A = E$ or $A = E_1$, for $R'$ a nodal curve with $A \cdot R' = 1$. Therefore (c) also follows.

If $g = 22$, we similarly find that we can assume that $A_1$ and $A_2$ are nef. Moreover $A_1 \cdot L_1 = A_1 \cdot (H - 2E) = E_1 \cdot (H - 2E) = 4$, so if $E_1$ is
not nef, we can substitute $E_1$ with $A_1$ and repeat the process. Therefore we can assume that $E_1$ is nef as well. Again both $|E_1 + A_1|$ and $|E + A_2|$ are without fixed components, and setting $F_1 = E_1$, $F_2 = A_2$, $F_3 = A_1$ and $F_4 = A_3$ we therefore have the desired decomposition satisfying (a) and (b). Now $E + F_1 + F_2 + F_3$ is again nef by construction. To see that $E + F_1 + F_4$ is nef, assume, to get a contradiction, that there is a nodal curve $\Gamma$ with $\Gamma \cdot (E + F_1 + F_4) < 0$. Then $\Gamma \cdot F_4 = -1$ and $\Gamma \cdot (E + F_1) = 0$ by (a). The ampleness of $H$ implies $\Gamma \cdot (F_2 + F_3) \geq 2$, whence the contradiction $(F_4 - \Gamma)^2 = 0$ and $(F_4 - \Gamma) \cdot (F_2 + F_3) \leq 0$, recalling that $F_4 - \Gamma > 0$ by [13, Lemma 2.3]. Therefore (c) is proved. We now prove (d).

If $g = 20$ then $(F_1 + F_4 - F_2)^2 = -2$ and $(F_1 + F_4 - F_2) \cdot H = 5 = \phi(H)$, whence $h^2(F_1 + F_4 - F_2) = 0$ and if $F_1 + F_4 - F_2 > 0$ it is a nodal cycle, so that either $h^0(F_1 + F_4 - F_2) = 0$ or $h^0(F_1 + F_4 - F_2 + K_S) = 0$. Replacing $F_1$ with $F_1 + K_S$ if necessary, we can arrange that $h^0(F_1 + F_4 - F_2) = 0$, whence also $h^1(F_1 + F_4 - F_2) = 0$ by Riemann-Roch.

If $g = 22$, then $(F_1 + F_4 - F_2)^2 = 0$ and $(F_1 + F_4 - F_2) \cdot H = 8 < 2\phi(H)$, whence (d) follows by Lemma 2.5 and [12, Cor. 2.5].

**Lemma 4.3.** In the cases $(E \cdot E_2, E_1 \cdot E_2, g) = (3, 2, 20)$ or $(3, 3, 22)$ in (5), $S$ is nonextendable.

**Proof.** By Claim 4.2 we can choose $D_0 = E + F_1 + F_2 + F_3$ with $D_0^2 = 12$ and both $D_0$ and $H - D_0 \sim E + F_1 + F_4$ base-point free. We have $h^0(2D_0 - H) = h^0(F_2 + F_3 - F_4) \leq 1$ by Lemma 2.5, as $(F_2 + F_3 - F_4) \cdot H \leq 6 < 2\phi(H)$. Hence the map $\Phi_{H-D_0} 2\phi(D_0)$ surjective by [14, Thm. (iii)-(iv)]. To show the surjectivity of $\mu_{V_D,op}$ we use Claim 4.2(b) and let $D_1 \in [E + F_2]$ and $D_2 \in [F_1 + F_3]$ be general smooth curves and apply [11, Lemma 5.6]. Now $H - D_0 - D_1 \sim F_1 + F_4 - F_2$ whence $h^1(H - D_0 - D_1) = 0$ by Claim 4.2(d), so that $\mu_{V_{D_1},op}$ is surjective by [11, (14)] since $(H - D_0) \cdot D_1 = (E + F_1 + F_4) \cdot (E + F_2) = 5$. Since $(H - D_0 - D_2) \cdot H = (E + F_3 - F_4) \cdot H \leq 7 < 2\phi(H)$ we have that $h^0(H - D_0 - D_2) \leq 1$ by Lemma 2.5 and $\mu_{V_{D_2},op}(D_2)$ is surjective by [11, (16)]. Therefore $\mu_{V_D,op}$ is surjective whence $S$ is nonextendable by [11, Prop. 5.1].

**Lemma 4.4.** If $E \cdot E_2 = 2$ and $(E_1 \cdot E_2, g) = (2, 17)$ or $(3, 19)$ in (5), then $S$ is nonextendable.

**Proof.** We first observe that it is enough to find an isotropic divisor $F > 0$ such that $E \cdot F = 1, F \cdot H = 6$ if $g = 17$ and $F \cdot H = 7$ if $g = 19$ and $B := E + F$ is nef. In fact the latter implies that $H \sim 2B + A$, with $A > 0$ isotropic with $E \cdot A = 2$ and $F \cdot A = 4$ if $g = 17$ and $F \cdot A = 5$ if $g = 19$. As $H$ is not 2-divisible in Num $S$, $A$ is automatically primitive and it follows that $S$ is nonextendable by [11, Lemma 5.5(iii-b)].
To find the desired $F$ we first consider the case $g = 17$.

Set $Q = E + E_1 + E_2$. Then $Q^2 = 10$ and $\phi(Q) = 3$. By [6, Cor. 2.5.5] there is an isotropic effective 10-sequence $\{f_1, \ldots, f_{10}\}$ with $3Q \sim f_1 + \cdots + f_{10}$. Since $E \cdot Q = E_1 \cdot Q = 3$, we can assume that $f_1 = E$ and $f_2 = E_1$ and then $E_2 \cdot f_1 = 1$ for $i \geq 3$. We now claim that $E + f_1$ is not nef for at most one $i \in \{3, \ldots, 10\}$. Indeed, note that, for $i \geq 3$, we have $f_i \cdot H = 6 < 2\phi(H) = 8$, whence each $f_i$ is quasi-nef by Lemma 2.5. Now assume that $R_i \cdot (E + f_i) < 0$ for some nodal curve $R_i$. Then $R_i \cdot E = 0$ and $R_i \cdot f_i = -1$, so that $f_i \sim \overline{f_i} + R_i$, by [13, Lemma 2.3], with $\overline{f_i} > 0$ primitive and $\overline{f_i}^2 = 0$. Since $H$ is ample we must have $R_i \cdot E_j > 0$ for $j = 1$ or 2. If $R_i \cdot E_2 > 0$ then $E_2 \cdot f_1 = 1$ implies $\overline{f_i} \equiv E_2$ and $R_i \cdot E_2 = 1$. But then we get the contradiction $E \cdot f_i = E \cdot (E_2 + R_i) = 2$. Therefore $R_i \cdot E_1 > 0$, so that $\overline{f_i} \equiv E_1$ and $R_i \cdot E_1 = 1$. Now suppose that also $E + f_i$ is not nef for $j \not\in \{3, \ldots, 10\} - \{i\}$. Then $R_i \cdot R_j = (f_i - E_1) \cdot (f_j - E_1) = -1$, a contradiction. Therefore $E + f_i$ is not nef for at most one $i \in \{3, \ldots, 10\}$. Now one easily verifies that any $F \in \{f_3, \ldots, f_{10}\}$ such that $E + F$ is nef satisfies the desired numerical conditions.

We next consider the case $g = 19$.

Since $(E_1 + E_2)^2 = 6$ and $\phi(E_1 + E_2) = 2$ we can find an isotropic effective 3-sequence $\{f_3, f_4, f_5\}$ such that $E_1 + E_2 \sim f_3 + f_4 + f_5$. Since $E \cdot (E_1 + E_2) = E_1 \cdot (E_1 + E_2) = 3$ we have $f_i \cdot E = f_i \cdot E_1 = 1$ for $i = 3, 4, 5$, so that we have an isotropic effective 5-sequence $\{f_3, \ldots, f_5\}$ with $f_1 = E$ and $f_2 = E_1$ such that $H \sim 3f_1 + f_2 + f_3 + f_4 + f_5$. By [6, Cor. 2.5.6] we can complete the sequence to an isotropic effective 10-sequence $\{f_1, \ldots, f_{10}\}$. Note that for $i \geq 6$ we have $f_i \cdot H = 7 < 2\phi(H) = 8$, whence each $f_i$ is quasi-nef by Lemma 2.5. Now the same arguments as above can be used to prove that $E + f_i$ is nef for at least one $i \in \{6, \ldots, 10\}$, whence any $F \in \{f_6, \ldots, f_{10}\}$ such that $E + F$ is nef satisfies the desired numerical conditions.

\[\blacksquare\]

4.1.3 $\beta = 4$

From (1) and (2) we get $1 \leq E \cdot E_2 \leq E_1 \cdot E_2 + 2 \leq E \cdot E_2 + a$.

If $a = 2$ we get $E \cdot E_2 - 2 \leq E_1 \cdot E_2 \leq E \cdot E_2$. Moreover, since $L_2 \sim 2E + E_2$ is not of small type, we get $\phi(L_2) = (E \cdot E_2)^2 \leq L_2^2 = 4E \cdot E_2$, whence $E \cdot E_2 \leq 3$ by [13, Prop. 1]. Therefore $(E \cdot E_2, E_1 \cdot E_2) \in \{(1, 1), (2, 1), (2, 2), (3, 1), (3, 2), (3, 3)\}$. The first case is case (ii) in Theorem 1.1 and in the other cases $S$ is nonextendable by (3) and [11, Lemma 5.5(iii-a)].

If $a = 3$ or 4 we must have $E_1 \cdot (H - aE) = \phi(H)$ by [11, (11)], whence $E_1 \cdot E_2 = E \cdot E_2 + a - 2$. Moreover $L_1 \sim (4 - a)E + 2E_1 + E_2$ and using $\phi(L_1)^2 \leq L_1^2$, we get $E_1 \cdot E_2 \leq 4$. If equality holds then $(L_2^2, \phi(L_1)) = (26, 5)$ or $(16, 4)$, both excluded by [13, Prop. 1], as $E_2$ is primitive. Therefore
We write $M_2 = E_2 + E_3$ as in Lemma 3.1(a).

4.2.1 $\beta = 2$

By Lemma 3.1 we have left to treat the cases (a-i), that is

$$(E \cdot E_2, E \cdot E_3, E_1 \cdot E_2, E_1 \cdot E_3) = (1, 2, 1, 2), (1, 2, 2, 1), (1, 1, 2, 2), (2, 2, 2, 2), (1, 2, 2, 2).$$

We first show that $S$ is nonextendable in the first case of (6).

Since $E_2 \cdot H = \phi(H) = 5$ and $E_3 \cdot H = 9 < 2\phi(H)$ we have that $E_2$ is nef and $E_3$ is quasi-nef by Lemma 2.5. In particular we get that $h^1(E_2 + E_3) = h^1(E_2 + E_3 + K_S) = 0$ by [12, Cor. 2.5] and $h^0(E_2 + E_3) = 2$ by Riemann-Roch. Now $D_0 := E + E_1 + E_2 + E_3$ is nef by [11, Lemma 6.3(b)] with $\phi(D_0) = 3$ and $D_0^2 = 16$. Also $H - D_0 \sim E + E_1$ is base-component free by (3) and $2D_0 - H \sim E_2 + E_3$. Then $h^0(2D_0 - H) = 2$ and $h^1(H - 2D_0) = 0$, so that $\mu_{V_D,opp}$ is surjective by [11, (13)] and $\Phi_{H_D,opp}$ is surjective by [14, Thm. (v)], as $\text{gon}(D) = 6$ by [13, Cor. 1], whence $\text{Cliff}(D) = 4$, as $D$ has genus 9 [7, §5]. By [11, Prop. 5.1], $S$ is nonextendable.

We next show that $S$ is nonextendable in the last four cases in (6).

By Lemma 2.5 and [11, Lemma 6.3(b)] we see that $E_2$ and $E_3$ are quasi-nef and $E + E_1 + E_2$ and $E + E_1 + E_3$ are base-point free. Set $D_0 = E + E_1 + E_2$.

Then $D_0^2 \geq 8$, $D_0$ is nef, $\phi(D_0) \geq 2$ and $H - D_0 \sim E + E_1 + E_3$ is base-point free. Moreover $h^0(2D_0 - H) = 0$ as $(2D_0 - H) \cdot H = (E_2 - E_3) \cdot H \leq 0$, so that $\Phi_{H_D,opp}$ is surjective by [14, Thm. (iii)]. Now, in all cases except for $(E \cdot E_2, E \cdot E_3, E_1 \cdot E_2, E_1 \cdot E_3) = (1, 2, 2, 2)$, we have $(H - 2D_0)^2 = -2$ and $(H - 2D_0) \cdot H = 0$, so that $h^0(H - 2D_0) = h^1(H - 2D_0) = 0$, whence $h^0(H - 2D_0) = 0$ by Riemann-Roch and $\mu_{V_D,opp}$ is surjective by [11, (12)] (noting that $(H - D_0)^2 = 10$ in the case $(2, 2, 2, 2)$, while $H - D_0$ is not 2-divisible in Pic $S$ as either $E \cdot (H - D_0) = 3$ or $E_1 \cdot (H - D_0) = 3$ in the other two cases). By [11, Prop. 5.1], $S$ is nonextendable in those cases.

We now prove the surjectivity of $\mu_{V_D,opp}$ in the case $(E \cdot E_2, E \cdot E_3, E_1 \cdot E_2, E_1 \cdot E_3) = (1, 2, 2, 2)$.

Note that $E_1 + E_2$ is nef by [11, Lemma 6.3(e)], whence base-point free, and that $E_1 + E_3$ is quasi-nef. To see the latter, let $\Delta > 0$ be such that $\Delta^2 = -2$ and $\Delta \cdot E_1 + \Delta \cdot E_3 \leq -2$. As $E_1$ is quasi-nef by (3) and $E_3$ is quasi-nef we get, again by (3), that $\Delta \cdot E_1 = \Delta \cdot E_3 = -1$ and $E_1 \equiv E + \Delta$, giving the contradiction $\Delta \cdot E_3 = 0$. Hence $E_1 + E_3$ is quasi-nef. To show the surjectivity
of $\mu_{V_{D,0}}$, we let $D_1 = E$ and $D_2 \in |E_1 + E_2|$ be a general smooth curve and apply [11, Lemma 5.6]. The map $\mu_{V_{D_1,0}}$ is surjective by [11, (15)] since $h^1(H - D_0 - D_1) = h^1(E_1 + E_3) = 0$ by [12, Cor. 2.5]. Finally, $\mu_{V_{D_2,0}}(D_1)$ is surjective by [11, (16)], using the fact that $h^0(H - D_0 - D_2) = h^0(E + E_3 - E_2) \leq 1$ by Lemma 2.5, as $(E + E_3 - E_2) \cdot H = 7 < 2\phi(H)$. Therefore $\mu_{V_{D,0}}$ is surjective and $S$ is nonextendable by [11, Prop. 5.1].

4.2.2 $\beta = 3, 4$

By Lemma 3.1 we have left to treat the cases (a-ii) and (a-iii), that is

$$\beta = 3, \quad (E \cdot E_2, E \cdot E_3, E_1 \cdot E_2, E_1 \cdot E_3, E_2 \cdot E_3) = (2, 2, 2, 2, 1), \quad (7)$$

$$\beta = 3, \quad (E \cdot E_2, E \cdot E_3, E_1 \cdot E_2, E_1 \cdot E_3, E_2 \cdot E_3) = (2, 2, 1, 2, 1), \quad (8)$$

$$\beta = 3, 4, \quad (E \cdot E_2, E \cdot E_3, E_1 \cdot E_2, E_1 \cdot E_3, E_2 \cdot E_3) = (1, 1, 2, 2, 1), \quad (9)$$

$$\beta = 3, 4, \quad (E \cdot E_2, E \cdot E_3, E_1 \cdot E_2, E_1 \cdot E_3, E_2 \cdot E_3) = (1, 1, 1, 2, 1), \quad (10)$$

$$\beta = 3, 4, \quad (E \cdot E_2, E \cdot E_3, E_1 \cdot E_2, E_1 \cdot E_3, E_2 \cdot E_3) = (1, 1, 1, 1, 1). \quad (11)$$

Claim 4.5. In the cases (7)-(11) both $E_2$ and $E_3$ are quasi-nef.

Proof. We first prove that $E_2$ is quasi-nef. Assume, to get a contradiction, that there exists a $\Delta > 0$ with $\Delta^2 = -2$ and $\Delta \cdot E_2 \leq -2$. By [13, Lemma 2.3] we can write $E_2 \sim A + k\Delta$, for $A > 0$ primitive with $A^2 = 0$ and $k = -\Delta \cdot E_2 = \Delta \cdot A \geq 2$. From $E_2 \cdot E_3 = 1$ it follows that $\Delta \cdot E_3 \leq 0$. If $\Delta \cdot E > 0$, we get from $2 \geq E \cdot E_2 = E \cdot A + kE \cdot \Delta$ that $E \cdot E_2 = k = 2, E \cdot \Delta = 1$ and $E \cdot A = 0$, whence the contradiction $E \equiv A$. Hence $\Delta \cdot E = 0$ and the ampleness of $H$ gives $\Delta \cdot E_1 \geq 2$ and the contradiction $E_1 \cdot E_2 = E_1 \cdot A + kE_1 \cdot \Delta \geq 4$. Hence $E_2$ is quasi-nef. The same reasoning works for $E_3$.

Lemma 4.6. $S$ is nonextendable in cases (7)-(9) and cases (10)-(11) with $\beta = 4$.

Proof. Define $D_0 = 2E + E_1 + E_2$, which is nef by [11, Lemma 6.3(a)] with $\phi(D_0) \geq 2$ and $D_0^2 \geq 12$ in cases (7)-(9) and $D_0^2 = 10$ in cases (10) and (11). Also $H - D_0 \sim (\beta - 2)E + E_1 + E_3$, whence $\phi(H - D_0) \geq 2$ and $H - D_0$ is base-point free by [11, Lemma 6.3(b)]. We have $2D_0 - H \sim (4 - \beta)E + E_2 - E_3$, whence $h^0(2D_0 - H) \leq 1$ in the cases (7)-(9), as $(2D_0 - H) \cdot H \leq \phi(H)$, and $h^0(2D_0 - H) = 0$ in cases (10)-(11), as
(2D₀ − H) · H ≤ 0. It follows from [14, Thm. (iii)–(iv)] that the map \( \Phi_{H,D,φ_D} \) is surjective.

We next note that \( \mu_{V_D,ω_D} \) is surjective by [11, (12)] if \( h^1(H - 2D₀) = h^1(E₃ - (4 - β)E - E₂) = 0 \).

Since \((E₃ - E₂) · H = 0 \) in cases (9) and (11) we have \( h^0(E₃ - E₂) = h^2(E₃ - E₂) = 0 \), whence \( h^1(E₃ - E₂) = 0 \) by Riemann-Roch. It follows that \( \mu_{V_D,ω_D} \) is surjective, whence \( S \) is nonextendable by [11, Prop. 5.1] in cases (9) and (11) with \( β = 4 \). In the remaining cases we can assume that

\[
h^1(E₃ - (4 - β)E - E₂) > 0.
\]

(12)

We next show that \( \mu_{V_D,ω_D} \) is surjective in case (8). For this we use (3), [11, Lemmas 5.6 and 6.3(c)] and let \( D₁ ∈ |E + E₁| \) and \( D₂ ∈ |E + E₂| \) be general smooth members.

By Claim 4.5 and [12, Cor. 2.5] we have that \( h^1(H - D₀ - D₁) = h^1(E₃) = 0 \), whence \( \mu_{V_D,ω_D₁} \) is surjective by [11, (14)]. Furthermore \( \mu_{V_D₂,ω_D₁} \) is surjective by [11, (16)], where one uses that \( h^0(H - D₀ - D₂) = h^0(E₁ + E₃ - E₂) ≤ 1 \) by Lemma 2.5 since \((E₁ + E₃ - E₂) · H < 2φ(H) \). Hence \( \mu_{V_D₂,ω_D₂} \) is surjective and \( S \) is nonextendable by [11, Prop. 5.1].

Finally we treat the cases (7), (9) (with \( β = 3 \)) and (10) (with \( β = 4 \)). Since \((E₃ - (4 - β)E - E₂)² = -2 \) and \((E₃ - (4 - β)E - E₂) · H = -φ(H) \) in (7) and (9) we see that Riemann-Roch and (12) imply that \( E + E₂ - E₃ + Kₕ \) is a nodal cycle in (7) and (9) and \( E₃ - E₂ \) is a nodal cycle in (10). With \( β \) as above, it follows that

\[
h^i(E + E₂ - E₃) = 0 \text{ in (7) and (9) and } h^i(E₃ - E₂ + Kₕ) = 0 \text{ in (10),}
\]

\[
i = 0, 1, 2.
\]

(13)

We now choose a new \( D₀ := (β - 2)E + E₁ + E₃ \), which is nef with \( φ(D₀) ≥ 2 \) and with \( H - D₀ \) base-point free by [11, Lemma 6.3(a)–(b)]. Then \( D₀² ≥ 8 \) with \( h^0(2D₀ - H) = h^0(E₃ - E₂) = 0 \) in (7) and (9) and \( D₀² = 12 \) with \( h^0(2D₀ - H) = h^0(E₃ - E₂) = 1 \) in (10), whence \( Φ_{H,D,φ_D} \) is surjective by [14, Thm. (iii)–(iv)]. Now (13) implies \( h^1(H - 2D₀) = 0 \), so that \( \mu_{V_D,ω_D} \) is surjective by [11, (12)] and \( S \) is nonextendable by [11, Prop. 5.1].

We have left the cases (10) and (11) with \( β = 3 \), which we treat in Lemmas 4.7 and 4.9.

**Lemma 4.7.** \( S \) is nonextendable in case (10) with \( β = 3 \).

**Proof.** Since \( E₂ · H = 6 \) one easily finds another decomposition of the same type

\[
H ∼ 3E + 2E₂ + E₁ + E₃', \quad \text{with } E₂ · E₃' = 2,
\]

(14)

and all other intersections equal to one.
We first claim that either $E_1$ or $E_2$ is nef. In fact $\phi(L_1) = E_1 \cdot L_1 = E_1 \cdot (E + 2E_1 + E_2 + E_3) = 4 = E_2 \cdot L_1$. By (3), if neither $E_1$ nor $E_2$ are nef, there are two nodal curves $R_1$ and $R_2$ such that $R_i \cdot E = 1$ and $E_i \equiv E + R_i$, for $i = 1, 2$. But then we get the absurdity $R_1 \cdot R_2 = (E_1 - E) \cdot (E_2 - E) = -1$.

By (14) we can and will from now on assume that we have a decomposition $H \sim 3E + 2E_1 + E_2 + E_3$ with $E_2$ nef.

**Claim 4.8.** Either $h^0(E + E_3 - E_2 - K_S) = 0$, or $h^0(E + E_2 - E_3) = 0$, or $h^0(E_2 + E_3 - E - K_S) = 0$.

**Proof.** Let $\Delta_i := E + E_3 - E_2 + K_S$, $\Delta_2 := E + E_2 - E_3$ and $\Delta_3 := E_2 + E_3 - E + K_S$. Assume, to get a contradiction, that $\Delta_i \geq 0$ for all $i = 1, 2, 3$. Since $\Delta_i^2 = -2$ we get that $\Delta_i > 0$ for all $i = 1, 2, 3$. We have $\Delta_2 \sim 2E + K_S - \Delta_1$. Since $\Delta_1 \cdot H = 6$ and $E \cdot H = 4$, we can neither have $\Delta_1 \leq E$ nor $\Delta_1 \leq E + K_S$. Therefore, as $E$ and $E + K_S$ have no common components, we must have $\Delta_1 = \Delta_{11} + \Delta_{12}$ with $0 < \Delta_{11} \leq E$ and $0 < \Delta_{12} \leq E + K_S$ and $\Delta_{11} \cdot \Delta_{12} = 0$. Moreover we have $E \cdot \Delta_{11} = E \cdot \Delta_{12} = 0$, whence $\Delta_{1i}^2 \leq 0$ for $i = 1, 2$. From $-2 = \Delta_1^2 = \Delta_{11}^2 + \Delta_{12}^2$ we must have $\Delta_{1i}^2 = 0$ either for $i = 1$ or for $i = 2$. By symmetry we can assume that $\Delta_{11}^2 = 0$. Therefore $\Delta_{11} \equiv qE$ for some $q \geq 1$ by [12, Lemma 2.1], but $\Delta_{11} \leq E$, whence $\Delta_{11} = E$ and $\Delta_{12}^2 = -2$. Moreover $\Delta_{12} \cdot H = 2$. Now since $E + \Delta_{12} \equiv \Delta_1 \equiv E + E_3 - E_2$, we get $E_3 \equiv E_2 + \Delta_{12}$ and $E_2 \cdot \Delta_{12} = 1$. Hence $\Delta_3 \sim E_2 + E_3 - E + K_S \sim (E + E_3 + K_S - \Delta_1) + 3 - E + K_S \sim 2E_3 - \Delta_1 \sim 2(E_2 + \Delta_{12}) - \Delta_1 \sim 2E_2 + \Delta_{12} - \Delta_{11}$, therefore

$$\Delta_{11} + \Delta_3 \in [2E_2 + \Delta_{12}].$$  

(15)

We claim that $|2E_2 + \Delta_{12}| = |2E_2| + \Delta_{12}$. To see the latter observe that it certainly holds if $\Delta_{12}$ is irreducible, for then it is a nodal curve with $E_2 \cdot \Delta_{12} = 1$ (recall that $|2E_2|$ is a genus one pencil). On the other hand if $\Delta_{12}$ is reducible then, using $\Delta_{12} \cdot H = 2$ and the ampleness of $H$ we deduce that $\Delta_{12} = R_1 + R_2$ where $R_1, R_2$ are two nodal curves with $R_1 \cdot R_2 = 1$. Moreover the nefness of $E_2$ allows us to assume that $E_2 \cdot R_1 = 1$ and $E_2 \cdot R_2 = 0$. But then $R_2 \cdot (2E_2 + \Delta_{12}) = -1$ so that $R_2$ is a base-component of $|2E_2 + \Delta_{12}|$ and of course $R_1$ is a base-component of $|2E_2 + \Delta_{12} - R_2| = |2E_2 + R_1|$ and the claim is proved.

Since $\Delta_{11}$ and $\Delta_{12}$ have no common components we deduce from (15) that each irreducible component of $E \sim \Delta_{11}$ must lie in some element of $|2E_2|$. The latter cannot hold if $E$ is irreducible for then we would have that $2E_2 - E > 0$ and $(2E_2 - E) \cdot E_2 = -1$ would contradict the nefness of $E_2$.

Therefore, as is well-known, we have that $E = R_1 + \cdots + R_n$ is a cycle of nodal curves and we can assume, without loss of generality, that $E_2 \cdot R_1 = 1$ and $E_2 \cdot R_i = 0$ for $2 \leq i \leq n$. As we said above, we have $2E_2 - R_1 > 0$. 
Now for $2 \leq i \leq n-1$ we get $R_i \cdot (2E_2 - R_1 - \cdots - R_{i-1}) = -1$, whence $2E_2 - R_1 - \cdots - R_i > 0$. Therefore $2E_2 - R_1 - \cdots - R_{n-1} > 0$ and since $R_n \cdot (2E_2 - R_1 - \cdots - R_{n-1}) = -2$ we deduce that $2E_2 - E > 0$, again a contradiction. \hfill \Box

**Conclusion of the proof of Lemma 4.7.** We divide the proof into the three cases of Claim 4.8.

**Case A.** $h^0(E + E_3 - E_2 + K_S) = 0$. Set $D_0 = 2E + E_1 + E_3$. Then $D_0^2 = 12$ and $\phi(D_0) = 2$. Moreover $D_0$ is nef by Claim 4.5 and [11, Lemma 6.3(a)] and $H - D_0 \sim E + E_1 + E_2$ is nef since $E + E_1$ and $E_2$ are (the first by (3)), so that $|H - D_0|$ is base-point free, since $\phi(H - D_0) = E \cdot (H - D_0) = 2$. We have $2D_0 - H \sim E + E_3 - E_2$ and since $(2D_0 - H) \cdot H = 6 < 2\phi(H) = 8$, we have $h^0(2D_0 - H) \leq 1$ by Lemma 2.5, so that $\Phi_{H_d,op}$ is surjective by [14, Thm. (iii)-(iv)]. Clearly $h^0(H - 2D_0) = 0$ and we also have $h^0(H - 2D_0) = h^0(2D_0 - H + K_S) = h^0(E + E_3 - E_2 + K_S) = 0$ by assumption. Therefore $h^1(H - 2D_0) = 0$ by Riemann-Roch and $\mu_{V_d,op}$ is surjective by [11, (12)]. Hence $S$ is nonextendable by [11, Prop. 5.1].

**Case B.** $h^0(E + E_2 - E_3) = 0$. We set $D_0 = E + E_1 + E_3$, so that $D_0^2 = 8$, $\phi(D_0) = 2$ and both $D_0$ and $H - D_0 \sim 2E + E_1 + E_2$ are nef by Claim 4.5 and [11, Lemma 6.3(a)-(b)], whence base-point free. Since $2D_0 - H \sim E_3 - E - E_2$ and $(E_3 - E - E_2) \cdot H < 0$ we have $h^0(2D_0 - H) = 0$, whence $\Phi_{H_d,op}$ is surjective by [14, Thm. (iii)]. Now by hypothesis $h^0(H - 2D_0) = 0$ and we also have $h^0(2D_0 - H + K_S) = h^0(E_3 - E - E_2 + K_S) = 0$, and by Riemann-Roch we get $h^1(H - 2D_0) = 0$ as well. Therefore $\mu_{V_d,op}$ is surjective by [11, (12)]. Hence $S$ is nonextendable by [11, Prop. 5.1].

**Case C.** $h^0(E_2 + E_3 - E + K_S) = 0$. Set $D_0 = E + E_1 + E_2 + E_3$, which is nef (since $E + E_1 + E_3$ is nef by Claim 4.5 and [11, Lemma 6.3(b)] and $E_2$ is nef by assumption) with $D_0^2 = 14$ and $\phi(D_0) = 3$. Moreover $H - D_0 \sim 2E + E_1$ is without fixed components. We have $H - 2D_0 \sim E - E_2 - E_3$ and since $(H - 2D_0) \cdot E = -2$ we have $h^0(E - E_2 - E_3) = 0$. By hypothesis we have $h^2(E - E_2 - E_3) = 0$, whence $h^1(H - 2D_0) = 0$ by Riemann-Roch. It follows that $\mu_{V_d,op}$ is surjective by [11, (12)]. Furthermore, since $2D_0 - H \sim E_2 + E_3 - E$ and $h^0(E_2 + E_3 - E + K_S) = 0$ we have $h^0(2D_0 - H) \leq 1$, and $\Phi_{H_d,op}$ is surjective by [14, Thm. (iii)-(iv)]. Hence $S$ is nonextendable by [11, Prop. 5.1]. \hfill \Box

**Lemma 4.9.** $S$ is nonextendable in case (11) with $\beta = 3$.

**Proof.** By Claim 4.5, [11, Lemma 6.3(d)] and symmetry, and adding $K_S$ to both $E_2$ and $E_3$ if necessary, we can assume that $|E + E_2|$ is base-component free.
Now set $D_0 = 2E + 2E_1 + E_3$. Then $D_0^2 = 16$ and $\phi(D_0) = 3$. Hence (3) and [11, Lemma 6.3(b)] give that $D_0$ is nef and $H - D_0 \sim E + E_2$ is base-component free. We have $H - 2D_0 \sim -(2E_1 + E_3 - E_2)$ and we now prove that $h^0(2D_0 - H) = 2$ and $h^1(H - 2D_0) = 0$. To this end, by [12, Cor. 2.5] and Riemann-Roch, we just need to show that $B$ and $\Delta_1$ are nef by [11, Lemma 6.3(e)] and $\Gamma_1$ is nef. Setting $\Delta_1 = \Delta_2$ and $\mu = \Delta_2$ give that $\Delta$ is nef by [11, Lemma 6.3(b)] and [11, Lemma 6.3(a)]. By symmetry between $B$ and $\Delta_1$, we can write $B \equiv B_0 + k\Delta$ where $k = -\Delta \cdot B \geq -2$. By [13, Lemma 2.3] we can write $B \equiv B_0 + k\Delta$ where $k = -\Delta \cdot B \geq 2$, $B_0 > 0$ and $B_0^2 = B^2 = 2$. Now $2 = E \cdot B = E \cdot B_0 + kE \cdot \Delta \geq 1 + 2E \cdot \Delta$, therefore $E \cdot \Delta = 0$. The ampleness of $H$ implies that $E_2 \cdot \Delta \geq 2$, giving the contradiction $4 = E_2 \cdot B = E_2 \cdot B_0 + kE_2 \cdot \Delta \geq 5$. Therefore $B$ is quasi-nef.

Let $D \in |D_0|$ be a general curve. By [13, Cor. 1] we know that $\text{gon}(D) = 2\phi(D_0) = 6$ whence $\text{Cliff}(D) = 4$, as $D$ has genus 9 [7, §5]. Therefore the map $\Phi_{H_0, \text{org}}$ is surjective by [14, Thm. (v)]. Also $\mu_{V_{1, \text{org}}}$ is surjective by [11, (13)] and $S$ is nonextendable by [11, Prop. 5.1].

4.3 The case $M_2^2 = 4$

We write $M_2 = E_2 + E_3$ as in Lemma 3.1(b).

4.3.1 $\beta = 2$

By Lemma 3.1 we have $(E \cdot E_2, E \cdot E_3) = (1, 2)$ and the four cases $(E_1 \cdot E_2, E_1 \cdot E_3) = (1, 2), (2, 1), (2, 2)$ and $(1, 3)$. Note that in all cases $E_2 \cdot H < 2\phi(H) = 10$, whence $E_2$ is quasi-nef by Lemma 2.5.

If $(E_1 \cdot E_2, E_1 \cdot E_3) = (1, 2)$ we claim that either $E + E_2$ or $E_1 + E_2$ is nef. Indeed if there is a nodal curve $\Gamma$ such that $\Gamma \cdot (E + E_2) < 0$ then $\Gamma \cdot E_2 = -1$ and $\Gamma \cdot E = 0$. By [11, Lemma 6.3(a)] we have $\Gamma \cdot E_1 > 0$, so that $E_2 \equiv E_1 + \Gamma$ and $E_1 + E_2 \equiv 2E_1 + \Gamma$ is nef. By symmetry the same arguments work if there is a nodal curve $\Gamma$ such that $\Gamma \cdot (E_1 + E_2) < 0$ and the claim is proved.

By symmetry between $E$ and $E_1$ we can now assume that $E + E_2$ is nef. Setting $A := H - 2E - 2E_2$ we have $A^2 = 0$. As $E \cdot A = 3$ and $E_2 \cdot A = 4$ we have that $A > 0$ is primitive and $S$ is nonextendable by [11, Lemma 5.5(iii-b)].

If $(E_1 \cdot E_2, E_1 \cdot E_3) = (2, 1)$ one easily sees that $H \sim 2(E_1 + E_2) + A$, with $A^2 = 0$, $E_1 \cdot A = 1$ and $E_2 \cdot A = 4$. Then $A > 0$ is primitive, $E_1 + E_2$ is nef by [11, Lemma 6.3(e)] and $S$ is nonextendable by [11, Lemma 5.5(ii)].

If $(E_1 \cdot E_2, E_1 \cdot E_3) = (1, 3)$ we have $(E_1 + E_3)^2 = 6$ and we can write $E_1 + E_3 \sim A_1 + A_2 + A_3$ with $A_i > 0$, $A_i^2 = 0$ and $A_1 \cdot A_j = 1$ for $i \neq j$. Then $E \cdot A_i = E_1 \cdot A_i = E_2 \cdot A_i = E_3 \cdot A_i = 1$ and $A_i \cdot H = 6$.

We now claim that either $A_i$ is nef or $A_i \equiv E + \Gamma_i$ for a nodal curve $\Gamma_i$ with $\Gamma_i \cdot E = 1$. In particular, at least two of the $A_i$’s are nef. If there is a
noidal curve $\Gamma$ with $\Gamma \cdot A_1 < 0$, then since $A_1 \cdot L_1 = 4 = \phi(L_1)$ we must have $\Gamma \cdot L_1 \leq 0$, whence $\Gamma \cdot E > 0$ by the ampleness of $H$ and the first statement immediately follows. If two of the $A_i$’s are not nef, say $A_1 \equiv E + \Gamma_1$ and $A_2 \equiv E + \Gamma_2$ then $1 = A_1 \cdot A_2 = (E + \Gamma_1) \cdot (E + \Gamma_2) = 2 + \Gamma_1 \cdot \Gamma_2$ yields the contradiction $\Gamma_1 \cdot \Gamma_2 = -1$ and the claim is proved.

We can therefore assume that $A_1$ and $A_2$ are nef. Let $A = H - 2A_1 - 2A_2$. Then $A^2 = 0$ and $E \cdot A = 1$, whence $A > 0$ is primitive. As $A_1 \cdot A = A_2 \cdot A = 4$ and $\phi(H) = 5$, we have that $S$ is nonextendable by [11, Lemma 5.5(iii-b)].

If $(E_1 \cdot E_2, E_1 \cdot E_3) = (2, 2)$, note first that $E_1 + E_2$ is nef by [11, Lemma 6.3(e)]. Set $A := H - 2E_1 - 2E_2$. Then $A^2 = 0$ and $A \cdot E = 1$, so that $A > 0$ is primitive. As $(E_1 + E_2) \cdot A = 6$, we have that $S$ is nonextendable by [11, Lemma 5.5(ii)].

### 4.3.2 $\beta = 3$

By Lemma 3.1 we have $(E \cdot E_2, E \cdot E_3) = (1, 2)$ and $(E_1 \cdot E_2, E_1 \cdot E_3) = (1, 3)$ or $(2, 1)$.

We first show that $E_i$ is quasi-nef for $i = 2, 3$. We have $H \cdot E_2 \leq 2 < 2\phi(H) = 10$, whence $E_2$ is quasi-nef by Lemma 2.5. Now let $\Delta > 0$ be such that $\Delta^2 = -2$ and $\Delta \cdot E_3 \leq -2$. By [13, Lemma 2.3] we can write $E_3 \sim A + k\Delta$, for $A > 0$ primitive with $A^2 = 0, k = -\Delta \cdot E_3 = \Delta \cdot A \geq 2$.

If $\Delta \cdot E > 0$, from $E \cdot E_3 = E \cdot A + k\Delta \cdot E$ we get that $k = 2, \Delta \cdot E = 1$ and $E \cdot A = 0$, whence the contradiction $E \equiv A$. Hence $\Delta \cdot E = 0$. We get the same contradiction if $\Delta \cdot E_2 > 0$. Hence, by the ampleness of $H$ we must have $\Delta \cdot E_1 \geq 2$, but this gives the contradiction $E_1 \cdot E_3 = E_1 \cdot A + k\Delta \cdot E_1 \geq 4$.

Hence also $E_3$ is quasi-nef.

We now treat the case $(E_1 \cdot E_2, E_1 \cdot E_3) = (1, 3)$.

Let $D_0 = 2E + E_1 + E_2$. Then $D_0^2 = 16, \phi(D_0) = 2$ and $D_0$ and $H - D_0 \sim E + E_1 + E_3$ are base-point free by [11, Lemma 6.3(a)-(b)]. Moreover $2D_0 - H \sim E + E_2 - E_3$, and since $(2D_0 - H) \cdot E = -1$, we have $h^0(2D_0 - H) = 0$ and it follows from [14, Thm. (iii)] that the map $\Phi_{H_2,D_0}$ is surjective.

After possibly adding $K_S$ to both $E_2$ and $E_3$, we can assume, by (3) and [11, Lemma 6.3(c)], that the general members of both $|E + E_1|$ and $|E + E_2|$ are smooth irreducible curves. Let $D_1 \in |E + E_1|$ and $D_2 \in |E + E_2|$ be two such curves. By [12, Cor. 2.5] we have $h^1(H - D_0 - D_1) = h^1(E_3) = 0$, whence $\mu_{V_{E_1,E_2}}$ is surjective by [11, (14)].

We now claim that $h^0(E_1 + E_3 - E_2) \leq 2$. Indeed, assume that $h^0(E_1 + E_3 - E_2) \geq 3$. Then $|E_1 + E_3 - E_2| = |M| + G$, with $G$ the base-component and $|M|$ base-component free with $h^0(M) \geq 3$. If $M^2 = 0$, then $M \sim IP$, for an elliptic pencil $P$ and an integer $i \geq 2$. But then $14 = (E_1 + E_3 - E_2) \cdot H = (IP + G) \cdot H \geq \ell P \cdot H \geq 4\phi(H) = 20$, a contradiction. Hence $M^2 \geq 4$, but since $M \cdot H \leq (E_1 + E_3 - E_2) \cdot H = 14$, this contradicts the Hodge index theorem.
Therefore we have shown that $h^0(E_1 + E_3 - E_2) \leq 2$ and $\mu_{\mathcal{O}_D, \mathcal{O}_D(H)}$ is surjective by [11, (16)]. By [11, Lemma 5.6], $\mu_{\mathcal{O}_D, \mathcal{O}_D}$ is surjective and by [11, Prop. 5.1], $S$ is nonextendable.

Next we treat the case $(E_1 \cdot E_2 \cdot E_3) = (2, 1)$.

Let $D_0 = 2E + E_1 + E_3$. Then $D_0^2 = 14$, $\phi(D_0) = 3$ and $D_0$ and $H - D_0 \sim E + E_1 + E_2$ are base-point free by [11, Lemma 6.3(a)--(b)]. Moreover $2D_0 - H \sim E + E_1 - E_2$, and since $E + E_3$ is nef by [11, Lemma 6.3(c)] and $(2D_0 - H) \cdot (E + E_3) = (E + E_3 - E_2) \cdot (E + E_3) = 1$, we get that $h^0(2D_0 - H) \leq 1$. It follows from [14, Thm. (iii)--(iv)] that the map $\Phi_{\mathcal{O}_E, \mathcal{O}_E}$ is surjective. Let $D_1 \in |E + E_1|$ and $D_2 \in |E + E_3|$ be two general members. By [12, Cor. 2.5] we have that $h^1(H - D_0 - D_1) = h^1(E_2) = 0$, whence $\mu_{\mathcal{O}_{D_1}, \mathcal{O}_{D_1}(H-D_0), \mathcal{O}_{D_1}}$. Since $\mathcal{O}_{D_1}$ is a base-point free pencil we get that $\mathcal{O}_{D_1(D-H-D_0), \mathcal{O}_{D_1}}$ is surjective by the base-point free pencil trick because $\deg(O_{D_1}(H-D_0-D_1+K_S)) = 3$, whence $h^1(O_{D_1}(H-D_0-D_1+K_S)) = 0$. We have $(E_1 + E_2 - E_3) \cdot H = 5 = \phi(H)$, whence $h^0(E_1 + E_2 - E_3) \leq 1$ and $\mu_{\mathcal{O}_{D_2, \mathcal{O}_{D_2}(D_1), \mathcal{O}_{D_2}}}$ is surjective by [11, (16)]. By [11, Lemma 5.6], $\mu_{\mathcal{O}_E, \mathcal{O}_E}$ is surjective and, by [11, Prop. 5.1], $S$ is nonextendable.

### 4.4 The case $M_2^2 = 6$

By Lemma 3.1 we have $\beta = 2$ and $M_2^2 = E_2 + E_3 + E_4$ as in that lemma. We note that $E_1$, $E_2$ and $E_3$ are nef by Lemma 2.5 and $E_4$ is quasi-nef by the same lemma.

By the ampleness of $H$ it follows that $D_0 := E + E_1 + E_2 + E_3 + E_4$ is nef with $D_0^2 = 24$, $\phi(D_0) = 4$ and $H - D_0 \sim E + E_1$ is base-component free. Since $H - 2D_0 \sim -(E_2 + E_3 + E_4)$ we have $h^1(H - 2D_0) = 0$ by [12, Cor. 2.5] and $h^0(2D_0 - H) = 4$ by Riemann-Roch. Then $\mu_{\mathcal{O}_D, \mathcal{O}_D}$ is surjective by [11, (13)] and so is $\Phi_{\mathcal{O}_D, \mathcal{O}_D}$ by [14, Thm. (v)], since $\text{gon}(D) = 8$ by [13, Cor. 1], whence $\text{Cliff} D = 6$ by [7, §5], as $g(D) = 13$. Hence $S$ is nonextendable by [11, Prop. 5.1].

### References


On the proof of the genus bound for Enriques–Fano threefolds